Problem Set 15

8.2 34, 48, 51   8.3 1, 2, 5, 15

Series Handout 9   extra credit 8.2 52

9) a) Note that $\frac{1}{n+2} > \frac{1}{n+n}$ for $n > 2$ so

$\frac{1}{n+2} > \frac{1}{2n}$ for $n > 2$

so $\sum_{n=3}^{\infty} \frac{1}{n+2} > \sum_{n=3}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=3} \frac{1}{n}$

But $\sum \frac{1}{n}$ diverges because it is the harmonic series and $\sum \frac{1}{n+2}$ has infinitely many terms greater than the harmonic series so it diverges as well.

b) Note that $\frac{1}{\sqrt{n^2+10}} > \frac{1}{\sqrt{n^2+n^2}}$ for $n > 3$

But $\frac{1}{\sqrt{n^2+n^2}} = \frac{1}{\sqrt{2n^2}} = \frac{1}{n\sqrt{2}}$ so again

$\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2+10}} > \sum_{n=3}^{\infty} \frac{1}{n\sqrt{2}} = \frac{1}{\sqrt{2}} \sum_{n=3}^{\infty} \frac{1}{n}$ which diverges

so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+10}}$ must also diverge by the same logic as above.

c) $\sum_{n=}^{\infty} \frac{1}{n \ln n} = \quad + \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \cdots \frac{1}{n \ln n}$

note that $\frac{1}{n \ln n}$ is strictly decreasing

Now we want to group the terms in a very specific way. Letting $a_n = \frac{1}{n \ln n}$ then

$\sum_{n=2}^{\infty} a_n = 0 + a_2 + (a_3 + a_4) + (a_5 + \cdots + a_8) + (a_9 + \cdots + a_{16}) + \cdots$

$> 0 + a_2 + 2a_4 + 4a_8 + 8a_{16} + \cdots + 2^k a_{2^{k+1}}$

$> \frac{1}{2}(0 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots + 2^k a_{2^k})$

$> \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{1}{2^k \ln 2^k} \cdot 2^k \right) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{\ln 2^k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k \cdot \ln 2}$

$> \frac{1}{2\ln 2} \sum_{k=1}^{\infty} \frac{1}{k}$ ← but this is the harmonic series and diverges. Therefore

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ also diverges.

c) (continued) letting $a_n = \frac{1}{n(\ln n)^p}$

by the same logic as before we group the things.

In fact for a monotonically decreasing series

$a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ the series

$\sum_{n=1}^{\infty} a_n$ converges iff the series

$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots 2^k a_{2^k}$

converges.

We showed previously that if it diverges so does $\sum_{n=1}^{\infty} a_n$. The proof of the other direction uses the comparison test showing that $\sum_{n=1}^{\infty} a_n \leq \sum_{k=0}^{\infty} 2^k a_{2^k}$ so if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges so does $\sum_{n=1}^{\infty} a_n$.

We can apply this to $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ which converges if

$$\sum_{k=0}^{\infty} \frac{2^k}{2^k(\ln 2^k)^p} = \sum_{k=0}^{\infty} \frac{\cancel{2^k}}{\cancel{2^k}(k\ln 2)^p} = \sum_{k=0} \frac{1}{(\ln 2)^p k^p} = \frac{1}{(\ln 2)^p} \sum_{k=0} \frac{1}{k^p}$$

which converges for $p > 1$.

Extra credit:

apply what we used above. Then

$\sum_{n=1}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))}$ converges if $\sum_{k=0}^{\infty} \frac{2^k}{2^k \ln 2^k (\ln(\ln 2^k))}$ converges

$$= \sum_{k=0}^{\infty} \frac{1}{k\ln 2(\ln(k\ln 2))} = \frac{1}{\ln 2} \sum \frac{1}{k(\ln k + \ln\ln 2)}$$

which looks similar to $\sum_{n=1}^{\infty} \frac{1}{n\ln n}$ and diverges

To see this we can apply it again replacing the ks with 2^z and multiply the general term by 2^z. That yields

$$\frac{1}{\ln 2} \sum \frac{\cancel{2^z}}{2^z(\ln 2^z + \ln\ln 2)} = \frac{1}{(\ln 2)^2} \sum_{z=0}^{\infty} \frac{1}{z + \frac{\ln\ln 2}{\ln 2}}$$

But this diverges just as the harmonic series diverges, apply integral test.

34. $\sum_{n=0}^{\infty} 2^n(x+1)^n = \sum_{n=0}^{\infty} [2(x+1)]^n = \sum_{n=1}^{\infty} [2(x+1)]^{n-1}$ is a geometric series with $r = 2(x+1)$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|2(x+1)| < 1$ $\Leftrightarrow$ $|x+1| < \frac{1}{2}$ $\Leftrightarrow$ $-\frac{1}{2} < x+1 < \frac{1}{2}$ $\Leftrightarrow$ $-\frac{3}{2} < x < -\frac{1}{2}$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{1}{1-2(x+1)} = \dfrac{1}{-1-2x}$ or $\dfrac{-1}{2x+1}$.

48. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim_{n\to\infty} \dfrac{1}{a_n} \neq 0$, and so $\sum_{n=1}^{\infty} \dfrac{1}{a_n}$ is divergent by the Test for Divergence.

51. The partial sums $\{s_n\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n. Also, the sequence $\{s_n\}$ is bounded since $s_n \leq 1000$ for all n. So by Theorem 8.1.7, the sequence of partial sums converges, that is, the series $\sum a_n$ is convergent.

52. (a) $$\text{RHS} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} = \frac{f_n f_{n+1} - f_n f_{n-1}}{f_n^2 f_{n-1} f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{(f_{n-1} + f_n) - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{1}{f_{n-1} f_{n+1}} = \text{LHS}$$

(b) $$\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty} \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} \right) \quad \text{[from part (a)]}$$

$$= \lim_{n\to\infty} \left[\left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3} \right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4} \right) + \left(\frac{1}{f_3 f_4} - \frac{1}{f_4 f_5} \right) + \cdots + \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}} \right) \right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{f_1 f_2} - \frac{1}{f_n f_{n+1}} \right) = \frac{1}{f_1 f_2} - 0 = \frac{1}{1 \cdot 1} = 1 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

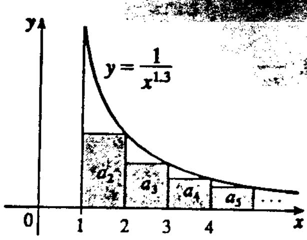
(c) $$\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty} \left(\frac{f_n}{f_{n-1}f_n} - \frac{f_n}{f_n f_{n+1}} \right) \quad \text{(as above)}$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}} \right)$$

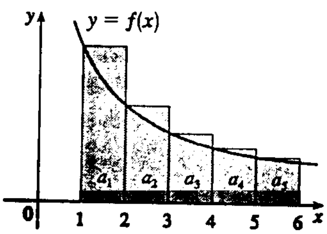
$$= \lim_{n\to\infty} \left[\left(\frac{1}{f_1} - \frac{1}{f_3} \right) + \left(\frac{1}{f_2} - \frac{1}{f_4} \right) + \left(\frac{1}{f_3} - \frac{1}{f_5} \right) + \left(\frac{1}{f_4} - \frac{1}{f_6} \right) + \cdots + \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}} \right) \right]$$

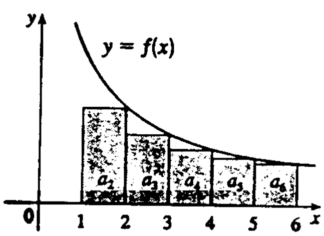
$$= \lim_{n\to\infty} \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}} \right) = 1 + 1 - 0 - 0 = 2 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

1. The picture shows that $a_2 = \dfrac{1}{2^{1.3}} < \displaystyle\int_1^2 \frac{1}{x^{1.3}}\,dx$, $a_3 = \dfrac{1}{3^{1.3}} < \displaystyle\int_2^3 \frac{1}{x^{1.3}}\,dx$, and so on, so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$. The integral converges by (5.10.2) with $p = 1.3 > 1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$.





5. $\sum_{n=1}^{\infty} n^b$ is a p-series with $p = -b$. $\sum_{n=1}^{\infty} b^n$ is a geometric series. By (1), the p-series is convergent if $p > 1$. In this case, $\sum_{n=1}^{\infty} n^b = \sum_{n=1}^{\infty} (1/n^{-b})$, so $-b > 1 \quad \Leftrightarrow \quad b < -1$ are the values for which the series converge. A geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$, so $\sum_{n=1}^{\infty} b^n$ converges if $|b| < 1 \quad \Leftrightarrow \quad -1 < b < 1$.

15. $f(x) = \dfrac{1}{x \ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\dfrac{1 + \ln x}{x^2(\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\int_2^{\infty} \dfrac{1}{x \ln x}\, dx = \lim_{t \to \infty} [\ln(\ln x)]_2^t = \lim_{t \to \infty} [\ln(\ln t) - \ln(\ln 2)] = \infty$, so the series diverges.