Problem Set 16

8-3 3, 4, 9, 10, 16, 18, 19, 20, 22

3. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note on page 587.)

(b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

4. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

9. $\dfrac{1}{n^2+n+1} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+n+1}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

10. $\dfrac{1}{2n-1} > \dfrac{1}{2n} = \dfrac{1}{2} \cdot \dfrac{1}{n}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n-1}$ diverges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2}\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, which diverges because it is a nonzero constant multiple of the divergent harmonic series.

16. $\dfrac{2}{n^3+4} < \dfrac{2}{n^3}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^3+4}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^3} = 2\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$, which converges because it is a constant multiple of a convergent p-series ($p = 3 > 1$).

18. $\dfrac{\sin^2 n}{n\sqrt{n}} \leq \dfrac{1}{n\sqrt{n}} = \dfrac{1}{n^{3/2}}$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges ($p = \frac{3}{2} > 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{\sin^2 n}{n\sqrt{n}}$ converges by the Comparison Test.

19. $\dfrac{n+1}{n^2} > \dfrac{n}{n^2} = \dfrac{1}{n}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n^2}$ diverges by comparison with the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$.

20. $\dfrac{4+3^n}{2^n} > \dfrac{3^n}{2^n} = \left(\dfrac{3}{2}\right)^n$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{4+3^n}{2^n}$ diverges by comparison with the divergent geometric series $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$.

22. Let $a_n = \dfrac{1}{n^3-n}$ and $b_n = \dfrac{1}{n^3}$. Then $\sum\limits_{n=2}^{\infty} a_n$ and $\sum\limits_{n=2}^{\infty} b_n$ are series with positive terms and

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3}{n^3-n} = 1 > 0$. Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^3}$ is a convergent p-series without the $n = 1$ term ($p = 3 > 1$), $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^3-n}$ is convergent by the Limit Comparison Test.