Math 1b, Solutions 22

#22 on Series Handout A

a) Recall: $\frac{d}{du}[\ln(1+u)] = \frac{1}{1+u} = 1-u+u^2-u^3+u^4-u^5+\cdots$

Integrate term-by-term the Maclaurin series for $\frac{1}{1+u}$ to get

the Maclaurin series for $\ln(1+u)$:

$$u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\cdots = \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}u^n}{n}}$$

b) Letting $u=x-1$, we have

$$\ln(1+u)=\ln(x) = (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots$$

$$= \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n}}$$

Series Handout A

24. (a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} +$

To get the 3rd degree polynomial for $f(x) = \sin x \cos x$, we only need to multiply the first two terms of the Taylor polynomials of $\sin x$ and $\cos x$.

$$\sin x \cos x \approx \left(x - \frac{x^3}{3!}\right)\left(1 - \frac{x^2}{2!}\right) = x - \frac{x^3}{2!} - \frac{x^3}{3!} \left[+ \frac{x^5}{3!2!} \right.$$

exclude because its degree is greater than 3.

$$= x - \frac{x^3}{2} - \frac{x^3}{6} = \boxed{x - \frac{2}{3}x^3}$$

Check:

$\sin(2x) = 2 \sin x \cos x$

$\frac{\sin(2x)}{2} = \sin x \cos x$

- substitute $2x$ for x in the Taylor polynomial for $\sin x$, up to the 3rd degree term

$$\frac{\sin(2x)}{2} = \frac{(2x) - \frac{(2x)^3}{3!}}{2} = \boxed{x - \frac{2}{3}x^3}$$

b) $\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1\cdot 3\cdot 5 \cdots (2n-3)}{2^n n!} x^{3n+1}$

To get the 2nd degree polynomial for $f(x) = \sqrt{1+x}\cdot\sqrt{1\ldots}$ we only need to multiply the 1st 3 terms

$$\Rightarrow \sqrt{1+x}\cdot\sqrt{1+x} = \left(1 + \frac{x}{2} - \frac{x^2}{8}\right)\cdot\left(1 + \frac{x}{2} - \frac{x^2}{8}\right)$$

$$= \boxed{1+x}$$ (excluding all terms with degree greater than 2)

- The x^3 coefficient should be 0 (remember that our function is just $1+x$)

Section 8.6 #9

$f(x) = \frac{1}{x-5} = -\frac{1}{5}\left(\frac{1}{1-\frac{x}{5}}\right) = -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$ or equivalently,

$-\sum_{n=0}^{\infty}\frac{1}{5^{n+1}}x^n$. The series converges when $\left|\frac{x}{5}\right| < 1$;

that is, when $|x| < 5$, so $I = (-5, 5)$.

11. (a) $f(x) = \frac{1}{(1+x)^2} = \frac{d}{dx}\left(\frac{-1}{1+x}\right) = -\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$= \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$ [from Theorem 2(a)] $= \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ with $R = 1$.

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \frac{1}{(1+x)^3} = -\frac{1}{2}\frac{d}{dx}\left[\frac{1}{(1+x)^2}\right] = -\frac{1}{2}\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R = 1$.

(c) $f(x) = \frac{x^2}{(1+x)^3} = x^2 \cdot \frac{1}{(1+x)^3} = x^2 \cdot \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$. To write the power series with x^n rather than x^{n+2},

we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$.

8.7)

11.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-1}	1
1	$-x^{-2}$	-1
2	$2x^{-3}$	2
3	$-3\cdot 2x^{-4}$	$-3\cdot 2$
4	$4\cdot 3\cdot 2x^{-5}$	$4\cdot 3\cdot 2$
$\vdots$	$\vdots$	$\vdots$

So $f^{(n)}(1)=(-1)^n n!$, and $\frac{1}{x}=\sum_{n=0}^{\infty}\frac{(-1)^n n!}{n!}(x-1)^n=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$. If $a_n=(-1)^n(x-1)^n$ then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|x-1|<1$ for convergence, so $R=1$.

29. $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $e^{-0.2}=\sum_{n=0}^{\infty}\frac{(-0.2)^n}{n!}=1-0.2+\frac{1}{2!}(0.2)^2-\frac{1}{3!}(0.2)^3+\frac{1}{4!}(0.2)^4-\frac{1}{5!}(0.2)^5+\frac{1}{6!}(0.2)^6-\cdots$.

But $\frac{1}{6!}(0.2)^6=8.\overline{8}\times 10^{-8}$, so by the Alternating Series Estimation Theorem, $e^{-0.2}\approx\sum_{n=0}^{5}\frac{(-0.2)^n}{n!}\approx 0.81873$, correct to five decimal places.

36. $\cos(x^2)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(x^2\right)^{2n}}{(2n)!}$, so

$\int_0^{0.5}\cos(x^2)\,dx=\int_0^{0.5}\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!}\,dx=\sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}\right]_0^{0.5}=0.5-\frac{(0.5)^5}{5\cdot 2!}+\frac{(0.5)^9}{9\cdot 4!}-\cdots$, but $\frac{(0.5)^9}{9\cdot 4!}\approx 0.000009$, so by the Alternating Series Estimation Theorem, $\int_0^{0.5}\cos(x^2)\,dx\approx 0.5-\frac{(0.5)^5}{5\cdot 2!}\approx 0.497$ (correct to three decimal places).