PROBLEM SET 23

$\boxed{8.7}$ **34.** $\int e^{x^3}\,dx = \int \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}\,dx = C + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)n!}$

$\boxed{8.8}$ **1.** The general binomial series in (2) is

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 +$$

$$\begin{aligned}(1+x)^{1/2} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n = 1 + \left(\tfrac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots \\ &= 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} + \frac{1 \cdot 3 \cdot x^3}{2^3 \cdot 3!} - \frac{1 \cdot 3 \cdot 5 \cdot x^4}{2^4 \cdot 4!} + \cdots \\ &= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)x^n}{2^n \cdot n!} \text{ for } |x| < 1, \text{ so } R = 1\end{aligned}$$

9. (a) $1/\sqrt{1-x^2} = \left[1 + (-x^2)\right]^{-1/2}$

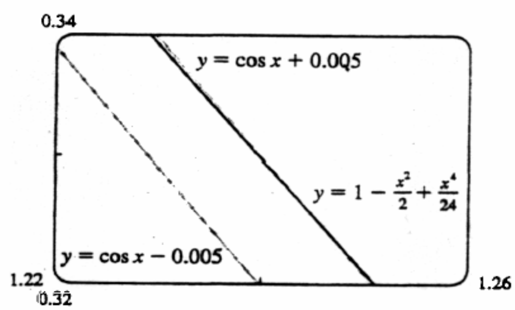
$$\begin{aligned}&= 1 + \left(-\tfrac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \\ &= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \cdot n!} x^{2n}\end{aligned}$$

(b) $\sin^{-1} x = \int \frac{1}{\sqrt{1-x^2}}\,dx = C + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1}$

$= x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1}$ since $0 = \sin^{-1} 0 = C$.

8.9)

22. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{6!}x^6\right| < 0.005 \Leftrightarrow x^6 < 720(0.005) \Leftrightarrow |x| < (3.6)^{1/6} \approx 1.238$. The curves intersect at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation are even functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.238 < x < 1.238$.



#23 on Series Handout A

$f(x) = x^{1/3}$

a) The first three derivatives of $f(x)$ are $f'(x) = \frac{1}{3}x^{-2/3}$

$$f''(x) = -\frac{2}{9}x^{-5/3}$$

$$f'''(x) = \frac{10}{27}x^{-8/3}$$

At $a = 27$, $T_3(x) = f(27) + f'(27)(x-27) + \frac{1}{2}f''(27)(x-27)^2 + \frac{1}{6}f'''(27)(x-27)^3$

$$T_3(x) = 3 + \left(\frac{1}{3}\right)\left(\frac{1}{9}\right)(x-27) - \left(\frac{1}{2}\right)\left(\frac{2}{9}\right)\left(\frac{1}{243}\right)(x-27)^2 + \left(\frac{1}{6}\right)\left(\frac{10}{27}\right)\left(\frac{1}{6561}\right)(x-27)^3$$

$$= \boxed{3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2 + \frac{5}{531441}(x-27)^3}$$

b) $T_3(28) = 3 + \frac{1}{27} - \frac{1}{2187} + \frac{5}{531441} = \boxed{\frac{1613768}{531441} \approx 3.036589198 \approx \sqrt[3]{28}}$

c) An upper bound for the error in this approximation is

$$E = |a_4| = \left|\underbrace{\left(\frac{1}{24}\right)}_{\frac{1}{6!}}\underbrace{\left(-\frac{8}{3}\right)\left(\frac{10}{27}\right)}_{\text{coefficient of derivative}}(27)^{-11/3}\underbrace{(1)}_{(28-27)^4}\right|$$

$$= \boxed{\frac{10}{43046721} \approx 0.0000002321}$$

Series Handout A

25. (a) A sum cannot be given because this geometric series diverges. It has $r = (-1)$, which is not included in the interval of convergence $-1 < r < 1$.

(b)
$$S = 3 + 3\cdot 7^2 + 3\cdot 7^4 + \cdots + 3\cdot 7^{20}$$
$$-\quad (7^2)\, S = \quad 3\cdot 7^2 + 3\cdot 7^4 + \cdots + 3\cdot 7^{20} + 3\cdot 7^{22}$$
$$(1-7^2)\, S = 3 - 3\cdot 7^{22}$$
$$S = \frac{3\,(1-7^{22})}{1-7^2}$$

Charlie's closed form formula, $\frac{a(1-r^{n+1})}{1-r}$, is correct. However, he calculated n wrong. The correct expression for the sum is $S = \frac{3(1-49^{11})}{1-49}$.

Charlie's statement about convergence is not correct. If this series were infinite, it would indeed diverge. But the series is finite, so it has a definite sum (that we calculated above).

(c) Charlie is incorrect. The Harmonic Series has $\lim_{n\to\infty} a_n$ and diverges. Amanda is referring to the n^{th} term test for divergence, which states that if $\lim_{n\to\infty} a_n \neq 0$, the series diverges. For $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.