PROBLEM SET 24

8.5

23. (a) If $a_n = \dfrac{(-1)^n x^{2n+1}}{n!(n+1)!\,2^{2n+1}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+1)!(n+2)!\,2^{2n+3}}\cdot\frac{n!(n+1)!\,2^{2n+1}}{x^{2n+1}}\right| = \left(\frac{x}{2}\right)^2 \lim_{n\to\infty}\frac{1}{(n+1)(n+2)} = 0$$ for all x. So $J_1(x)$ converges for all x and its domain is $(-\infty, \infty)$.

38. $$\int_0^{0.5} x^2 e^{-x^2}\,dx = \int_0^{0.5}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n!}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+3}}{n!(2n+3)}\right]_0^{1/2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+3)2^{2n+3}}$$ and since $c_2 = \frac{1}{1792} < 0.001$ we use $\sum_{n=0}^{1}\frac{(-1)^n}{n!(2n+3)2^{2n+3}} = \frac{1}{24} - \frac{1}{160} \approx 0.0354$.

23. Let $s(t)$ be the position function of the car, and for convenience set $s(0) = 0$. The velocity of the car is $v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$, so the second degree Taylor polynomial is $T_2(t) = s(0) + v(0)t + \frac{a(0)}{2}t^2 = 20t + t^2$. We estimate the distance travelled during the next second to be $s(1) \approx T_2(1) = 20 + 1 = 21$ m. The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of 2 m/s^2 for that long (if it did, its final speed would be 140 m/s $\approx$ 313 mi/h!)

(extra credit)

25. $$E = \frac{q}{D^2} - \frac{q}{(D+d)^2} = \frac{q}{D^2} - \frac{q}{D^2(1+d/D)^2} = \frac{q}{D^2}\left[1 \quad \left(1+\frac{d}{D}\right)^{-2}\right]$$

We use the Binomial Series to expand $(1 + d/D)^{-2}$:

$$E = \frac{q}{D^2}\left[1 - \left(1 - 2\left(\frac{d}{D}\right) + \frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2 \quad \frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3 + \cdot\right)\right]$$

$$= \frac{q}{D^2}\left[2\left(\frac{d}{D}\right) - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 \quad\right] \approx \frac{q}{D^2}\cdot 2\left(\frac{d}{D}\right) = 2qd\cdot\frac{1}{D^3}$$

Concept Check

3. (a) A geometric series has some r where $r = \frac{a_{n+1}}{a_n}$ for all n. An infinite geometric series converges when $|r|<1$ and converges to $S = \frac{a}{1-r}$

(b) A p-series is in the form $\sum \frac{1}{x^p}$ It converges when $p > 1$.

4. If $\sum a_n = 3$, then $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} s_n = 3$.

10. (a) $T_n(x) = \sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i$

True False

· see next page.

(b) $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

(c) $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ [$a = 0$ in part (b)]

(d) See Theorem 8.7.8.

(e) See Taylor's Inequality (8.7.9).

11. (a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdot$ $\quad(-1, 1)$

(b) $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x}{\cdot\cdot} + \frac{x^2}{2!} + \frac{x^3}{3!} +$ $\quad(-\infty, \infty)$

(c) $\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +$ $\quad(-\infty, \infty)$

(d) $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdot\cdot$ $\quad(-\infty, \infty)$

$\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdot\cdot$ $\quad[-1, 1]$

TRUE-FALSE

1. False. See Note 2 after Theorem 8.2.6.
2. True by Theorem 8.5.3.
   Or: Use the Comparison Test to show that $\sum c_n(-2)^n$ converges absolutely.
3. False. For example, take $c_n = (-1)^n / (n6^n)$.
4. True by Theorem 8.5.3.
5. False, since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{n^3}{(n+1)^3}\right| = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^3} = 1$.
6. True, since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{n!}{(n+1)!}\right| = \lim\limits_{n\to\infty}\dfrac{1}{n+1} = 0 < 1$.
7. False. See the note after Example 4 in Section 8.3.
8. True, since $\dfrac{1}{e} = e^{-1}$ and $e^x = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}$, so $e^{-1} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{n!}$.
9. True. See (6) in Section 8.1.
10. True, because if $\sum |a_n|$ is convergent, then so is $\sum a_n$ by Theorem 8.4.1.
11. True. By Theorem 8.7.5 the coefficient of x^3 is $\dfrac{f'''(0)}{3!} = \dfrac{1}{3} \Rightarrow f'''(0) = 2$.
    Or: Use Theorem 8.6.2 to differentiate f three times.
12. False. Let $a_n = n$ and $b_n = -n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n + b_n = 0$, so $\{a_n + b_n\}$ is convergent.
13. False. For example, let $a_n = b_n = (-1)^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n b_n = 1$, so $\{a_n b_n\}$ is convergent.
14. True by Theorem 8.1.7 (the Monotonic Sequence Theorem), since $\{a_n\}$ is decreasing and $0 < a_n \le a_1$ for all n $\Rightarrow$ $\{a_n\}$ is bounded.
15. True by Theorem 8.4.1. [$\sum (-1)^n a_n$ is absolutely convergent and hence convergent.]
16. True. $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n$ converges (Ratio Test) $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$ [Theorem 8.2.6].
17. False. The Integral Test tells us that the series $\sum_{n=1}^{\infty} a_n$ also converges, but its value is not equal to the value of $\int_1^{\infty} f(x)\,dx$. In fact, a picture like Figure 2 on page 584 shows that the sum of the series is larger than the value of the integral.

8.6

6. $f(x) = \frac{1}{1+9x^2} = \frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty}(-9x^2)^n = \sum_{n=0}^{\infty}(-1)^n 3^{2n} x^{2n}$. The series converges when $|-9x^2| < 1$; that is, when $|x| < \frac{1}{3}$, so $I = \left(-\frac{1}{3}, \frac{1}{3}\right)$.

16. From Example 7, $g(x) = \arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$. Thus,

$$f(x) = \arctan(x/3) = \sum_{n=0}^{\infty}(-1)^n \frac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{3^{2n+1}(2n+1)} x^{2n+1} \text{ for } \left|\frac{x}{3}\right| < 1 \quad \Leftrightarrow \quad |x| < 3,$$

so $R = 3$.