Diff Eq. Handout

PSET 27

6) a) $\frac{dM}{dt} = \text{rate}(M) = 0.04M$ Init. Cond: $t=0$ $M=2000$ $\left(\frac{dM}{dt}=80\right)$

b) $M = Ce^{0.04t}$ $M(0)=C=2000. \Rightarrow M(t) = 2000\,e^{0.04t}$

c) $\frac{dM}{dt} = 0.04M + 1000$ $M(0)=2000.$

d) $u = M + 25000,$ $\frac{du}{dt} = \frac{dM}{dt} = 0.04u \Rightarrow u = Ce^{0.04t},$ $M = Ce^{0.04t} - 25000$

$M = 27000\,e^{0.04t} - 25000$

7) $\frac{dB}{dt} = B\cdot r - 12000 = 0.0725B - 12000$

($B\cdot r$ ↑ interest accum.; 12000 ↑ amt. paid)

8) a) $\frac{dM}{dt} = kM$ $k = \frac{r}{M} = \frac{250}{5000} = \frac{1}{20}$ $\frac{dM}{dt} = \frac{M}{20}$

b) $\frac{dB}{dt} = kB$ $B = Ce^{kt}$ $B(0) = C = 600$

$B(10) = 600e^{10k} = 800 \Rightarrow k = \frac{\log(800/600)}{10} = \log(4/3)/10.$

$B(t) = 600e^{1/10\ln(4/3)t}$

9) a) $\frac{dP}{dt} = 0.03P - 6000$

b) $\frac{dP}{dt} = 0.03(P - 200{,}000)$

$u = P - 200{,}000$

$\frac{du}{dt} = \frac{dP}{dt} = 0.03u$

$u = Ce^{0.03t}$

$P = Ce^{0.03t} + 200{,}000.$ $P(0) = C + 200{,}000 = 3{,}000{,}000$

$\Rightarrow C = 2{,}800{,}000$

$P = 2{,}800{,}00\,e^{0.03t} + 200{,}000$

7. (a) Since the derivative $y' = -y^2$ is always negative (or 0 if $y = 0$), the function y must be decreasing (or equal to 0) on any interval on which it is defined.

(b) $y = \dfrac{1}{x+C} \quad \Rightarrow \quad y' = -\dfrac{1}{(x+C)^2}$. LHS $= y' = -\dfrac{1}{(x+C)^2} = -\left(\dfrac{1}{x+C}\right)^2 = -y^2 =$ RHS

(c) $y = 0$ is a solution of $y' = -y^2$ that is not a member of the family in part (b).

(d) If $y(x) = \dfrac{1}{x+C}$, then $y(0) = \dfrac{1}{0+C} = \dfrac{1}{C}$. Since $y(0) = 0.5$, $\dfrac{1}{C} = \dfrac{1}{2} \quad \Rightarrow \quad C = 2$, so $y = \dfrac{1}{x+2}$.

4. $y' = xy \quad \Rightarrow \quad \int \frac{dy}{y} = \int x\,dx \quad [y \neq 0] \quad \Rightarrow \quad \ln|y| = \frac{x^2}{2} + C \quad \Rightarrow \quad |y| = e^C e^{x^2/2} \quad \Rightarrow \quad y = Ke^{x^2/2}$

where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y =$ allowing K to be zero.)

10. $\frac{dy}{dx} = \frac{y\cos x}{1+y^2}$, $y(0) = 1$. $(1+y^2)\,dy = y\cos x\,dx \quad \Rightarrow \quad \frac{1+y^2}{y}dy = \cos x\,dx \quad \Rightarrow$

$\int\left(\frac{1}{y} + y\right)dy = \int \cos x\,dx \quad \Rightarrow \quad \ln|y| + \frac{1}{2}y^2 = \sin x + C$. $y(0) = 1 \quad \Rightarrow \quad \ln 1 + \frac{1}{2} = \sin 0 + C \quad \Rightarrow$

$C = \frac{1}{2}$, so $\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}$. We cannot solve explicitly for y.