Problem Set 34 25 supplement 16

25) a) $x'' = -bx' - cx$ we expect b and c to be positive constants, friction and restoring force.

b) $x'' + bx' + cx = 0$ with $b, c > 0$

roots $\dfrac{-b \pm \sqrt{b^2-4c}}{2}$

case 1 $b^2 - 4c > 0$ then roots

roots $r_1 = \dfrac{-b - \sqrt{b^2-4c}}{2}$, $r_2 = \dfrac{-b + \sqrt{b^2-4c}}{2}$

Since $c > 0$ $-b + \sqrt{b^2-4c} < 0$ so both roots negative.

solution $Ae^{r_1 t} + Be^{r_2 t} = y(t)$ with $r_1, r_2 < 0$ so

$\lim_{t\to\infty} y(t) = 0$

case 2 $b^2 - 4c = 0$ then one root $-\frac{b}{2}$

solution $Ae^{rt} + Bte^{rt} = y(t)$ with $r = -\frac{b}{2}$

and $\lim_{t\to\infty} y(t) = 0$ to see why

$\lim_{t\to\infty} te^{rt} = 0$ note $\lim_{t\to\infty} \dfrac{t}{\frac{1}{e^{rt}}} = \lim_{t\to\infty} \dfrac{1}{-re^{-rt}}$

$= \lim_{t\to\infty} \dfrac{1}{-r} e^{rt} = 0$ (use l'hopital)

case 3 $b^2 - 4c < 0$ then imaginary roots

$\dfrac{-b \pm \sqrt{b^2-4c}}{2}$ let $d = \sqrt{b^2-4c}$

Solution $y(t)\ e^{-\frac{b}{2}t}[A\cos(dt) + B\sin(dt)]$

and $\lim_{t\to\infty} y(t) = 0$ as $\lim_{t\to\infty} e^{-\frac{b}{2}t} = 0$

16) $mx'' = -kx$ $m = 2\,kg$

$10 = +k \cdot .1$ $k = 100$

then $2x'' + 100x = 0$

$x'' + 50x = 0$

general solution is $C_1 \cos 5\sqrt{2}t + C_2 \sin 5\sqrt{2}t$

$$x(0) = -2 = C_1 \cos 0 + C_2 \sin 0 = C_1$$
$$x'(t) = -5\sqrt{2}\, C_1 \sin 5\sqrt{2}t + 5\sqrt{2}\, C_2 \cos 5\sqrt{2}t$$
$$x'(0) = 0 = -5\sqrt{2}\, C_1 \sin 0 + 5\sqrt{2}\, C_2 \cos 0$$
$$0 = 5\sqrt{2}\, C_2 \rightarrow C_2 = 0$$

solution is $x(t) = -2 \cos 5\sqrt{2}\, t$

I. $y'' + 4y = 0$ $\quad r = \pm 2i$

solution $Ae^{2it} + Be^{-2it} = y(t)$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$y(0) = 2 = A + B$

$y'(0) = 0 = A - B$

$A = 1 \quad B = 1$

then $y(t) = e^{2it} + e^{-2it} = \sum_{n=0}^{\infty} \frac{(2it)^n + (-2it)^n}{n!}$

for n odd $(2it)^n + (-2it)^n = 0$

so for n even $(2it)^n + (-2it)^n = 2(2it)^n$

$$= \sum_{n=0}^{\infty} \frac{2(2it)^{2n}}{(2n)!} = 2\left(1 - \frac{2^2 t^2}{2!} + \frac{2^4 t^4}{4!} \cdots\right)$$

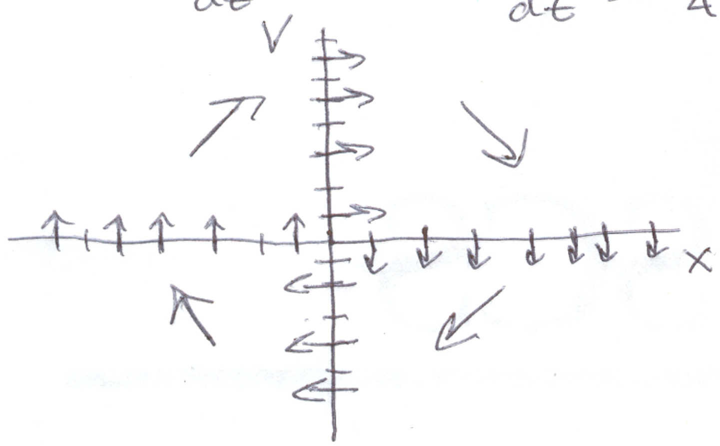
$$= 2 \cos(2t)$$

II a) $r = \pm\sqrt{-\frac{1}{4}} = \pm \frac{i}{2}$ solution $C_1 \cos \frac{t}{2} + C_2 \sin \frac{t}{2}$

$x(0) = 1 \rightarrow C_1 = 1$ $\quad x'(0) = 0 \rightarrow C_2 = 0$ so

particular solution $\cos \frac{t}{2}$

b) $\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\frac{1}{4}x$

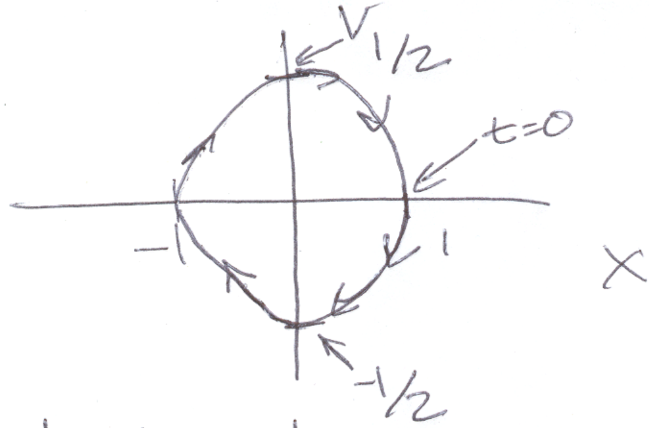


c) $\frac{dv}{dx} = \frac{-\frac{1}{4}x}{v} \Rightarrow v\,dv = -\frac{1}{4}x\,dx$

$$\frac{v^2}{2} = -\frac{1}{8}x^2 + C$$

which are ellipses, closed curves, yes this makes sense, since it should oscillate in the frictionless model.

d)



$\frac{dx}{dt} = v = -\frac{1}{2}\sin\frac{t}{2}$

so max of v is $\frac{1}{2}$.

e) $x'' + \frac{1}{10}x' + \frac{1}{4}x = 0$

roots $\frac{-\frac{1}{10} \pm \sqrt{\frac{1}{100} - 1}}{2} = -\frac{1}{20} \pm \frac{\sqrt{\frac{-99}{100}}}{2}$

$= -\frac{1}{20} \pm \frac{i\sqrt{99}}{20}$

$= -\frac{1}{20} \pm \frac{3\sqrt{11}\,i}{20}$

solution is $e^{-\frac{1}{20}t}\left[C_1\cos\frac{3\sqrt{11}}{20}t + C_2\sin\frac{3\sqrt{11}}{20}t\right]$

$x(0) = 1 \Rightarrow C_1 = 1$

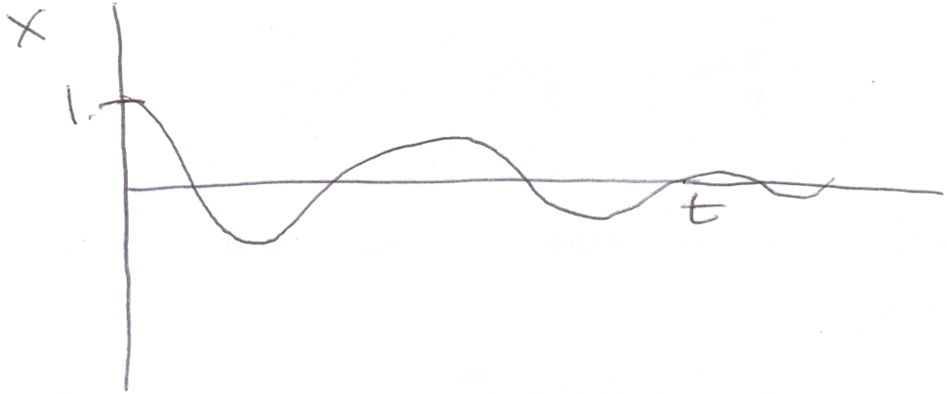
$x'(0) = 0 \Rightarrow C_2 = \frac{1}{3\sqrt{11}}$

$x'(t) = e^{-\frac{1}{20}t}\left[-\frac{3\sqrt{11}}{20}C_1\sin\frac{3\sqrt{11}}{20}t + \frac{3\sqrt{11}}{20}C_2\cos\frac{3\sqrt{11}}{20}t\right]$

$+ -\frac{1}{20}e^{-\frac{1}{20}t}\left[C_1\cos\frac{3\sqrt{11}}{20}t + C_2\sin\frac{3\sqrt{11}}{20}t\right]$

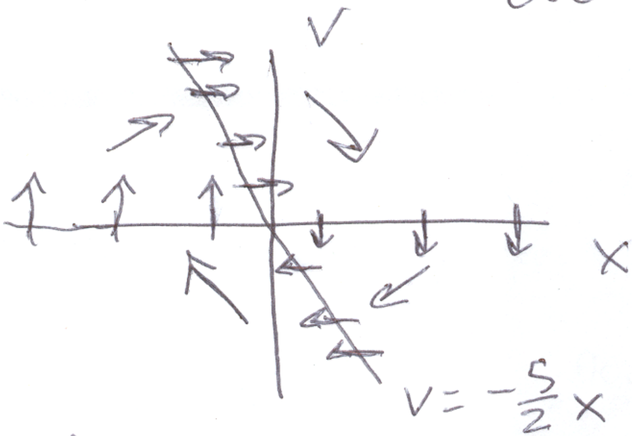
$x'(0) = 0 = \frac{3\sqrt{11}}{20}C_2 + \left(\frac{-1}{20}\right)C_1 \Rightarrow C_2 = \frac{1}{3\sqrt{11}}$

$x(t) = e^{-\frac{1}{20}t}\left(\cos\frac{3\sqrt{11}}{20}t + \frac{1}{3\sqrt{11}}\sin\frac{3\sqrt{11}}{20}t\right)$



f) $\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\frac{1}{10}v - \frac{1}{4}x$

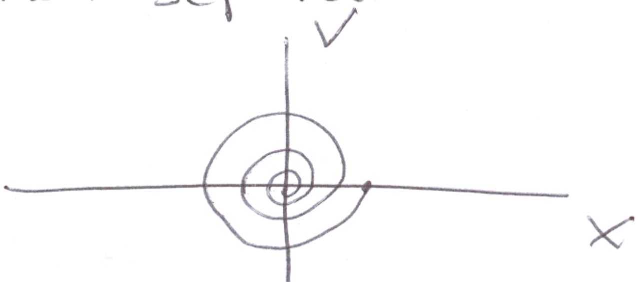
$v = -\frac{5}{2}x$



g) $\frac{dv}{dx} = \frac{-\frac{1}{10}v - \frac{1}{4}x}{v} = -\frac{1}{10} - \frac{1}{4}\frac{x}{v}$

not separable

h)



spiral in