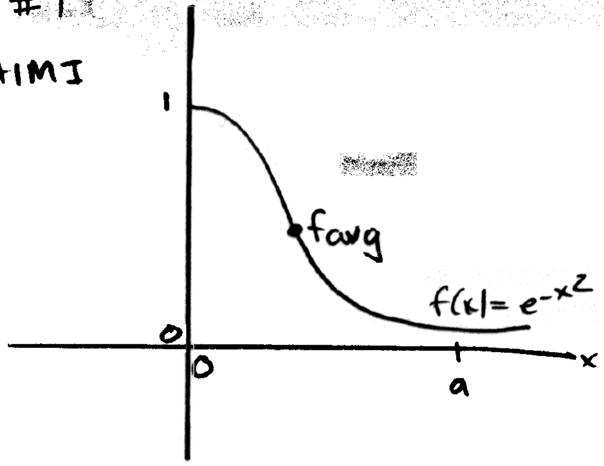
Integration HO #1 Problem Set 4

CLAIM I



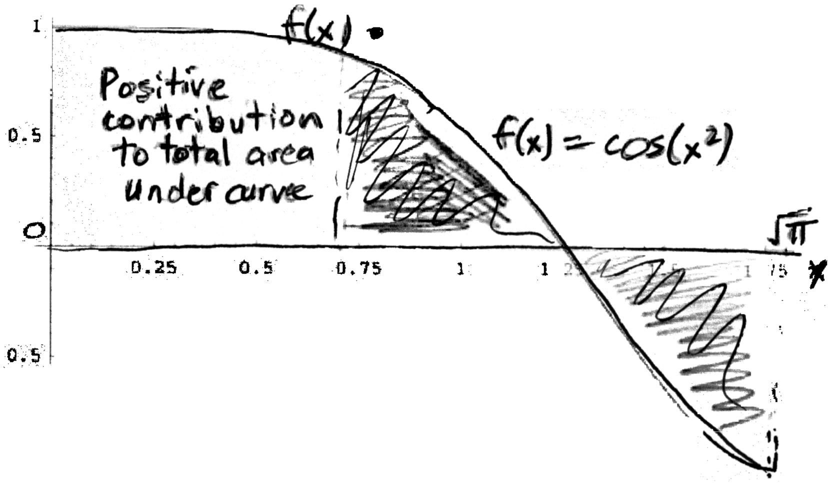
$0 < \int_0^a e^{-x^2}\,dx < a$
$0 < \underbrace{\frac{1}{a}\int_0^a e^{-x^2}\,dx}_{f_{avg}} < 1$

These are equivalent statements.

By definition, the average value of the function must lie somewhere between $f(x)=0$ and $f(x)=1$

$\boxed{\text{TRUE.}}$

CLAIM II:

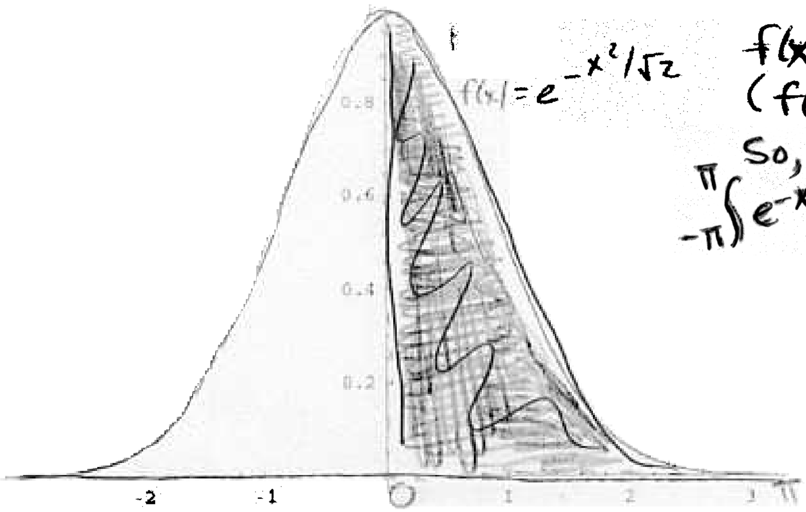


By inspection, we can see that the shaded areas are equal

$\int_0^{\sqrt{\pi}} \cos(x^2)\,dx > 0$

$\boxed{\text{TRUE}}$

CLAIM III



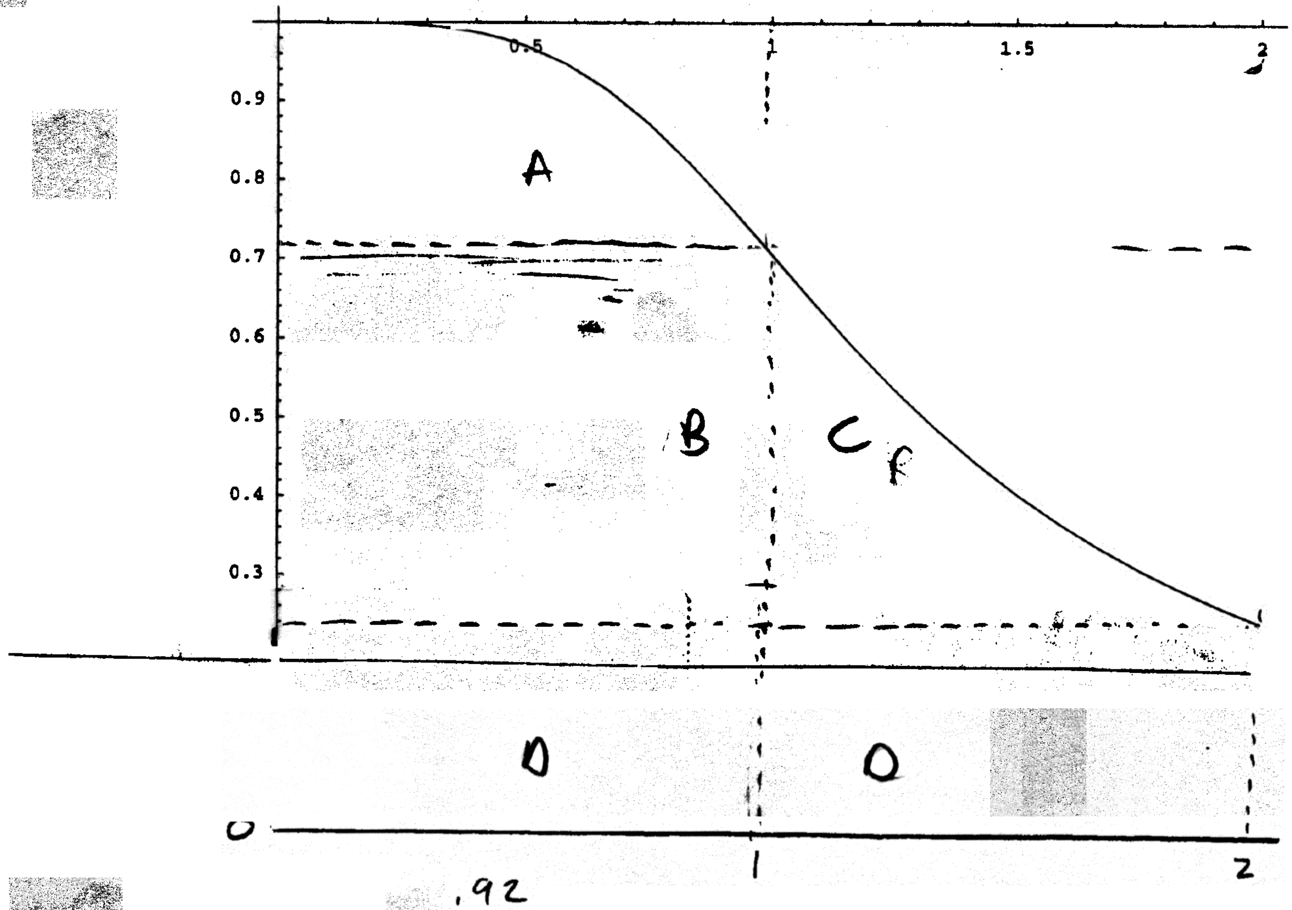
f(x) is even, by inspection ($f(-x) = f(x)$).

So,

$\int_{-\pi}^{\pi} e^{-x^2/\sqrt{2}}\,dx = 2\int_0^{\pi} e^{-x^2/\sqrt{2}}\,dx$

$\boxed{\text{TRUE}}$

CLAIM 4



$$\int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx = \text{Area}_A + \text{Area}_B + \text{Area}_D$$

$$\int_1^2 \frac{1}{\sqrt{1+x^4}}\,dx = \text{Area}_C + \text{Area}_D$$

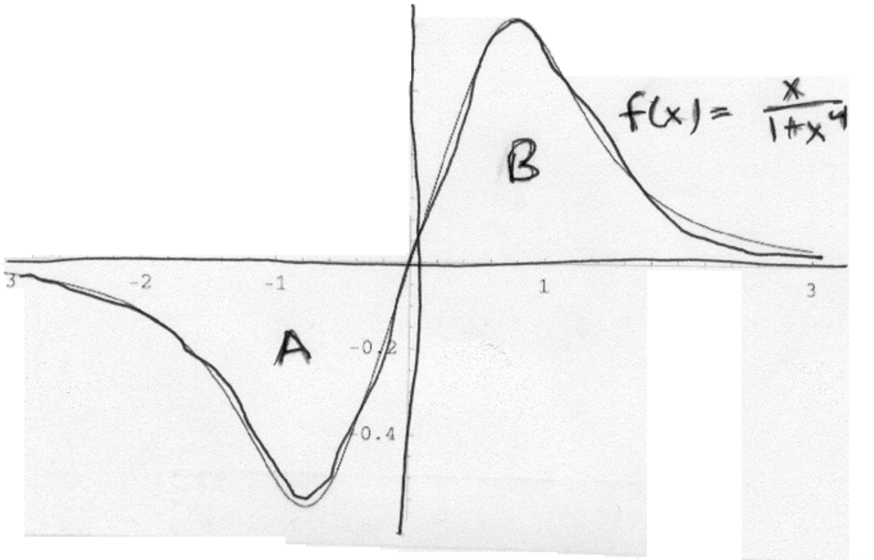
By inspection, Area B > Area C.

Compare $\text{Area}_A + \text{Area}_B$ and Area_C as Areas D are the same.

We know, $\text{Area}_B > \text{Area}_C \Rightarrow \text{Area}_A + \text{Area}_B$ must also be greater than Area_C (as Areas $A, B, C > 0$)

So, $\int_0^1 \frac{1}{\sqrt{1+x^4}} > \int_1^2 \frac{1}{\sqrt{1+x^4}}$ The claim was false

CLAIM 5



$f(x)$ appears to be odd, by inspection.

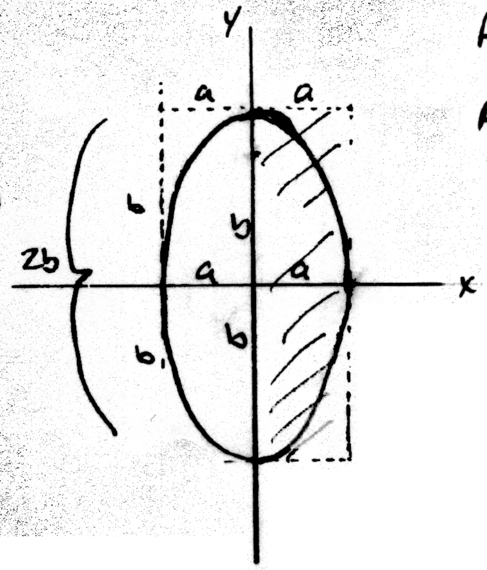
$[f(-x) = -f(x)]$ Both regims A and B appear to have the same area

$$\int_{0}^{3} \frac{x}{1+x^4}\,dx = -\int_{3}^{0} \frac{x}{1+x^4}\,dx \Rightarrow \int_{-3}^{3} \frac{x}{1+x^4}\,dx = 0 < 001$$

CLAIM is FALSE.

CLAIM 6

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



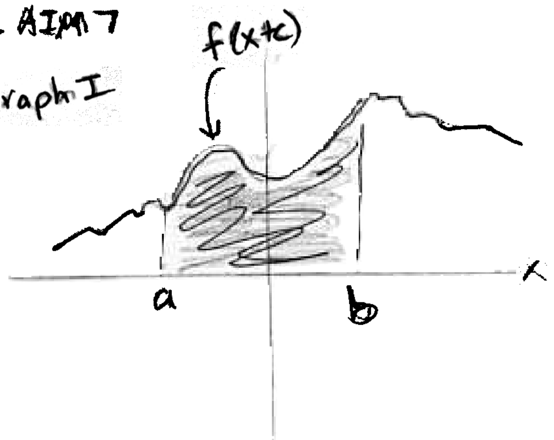
Area $\square = 2ab$

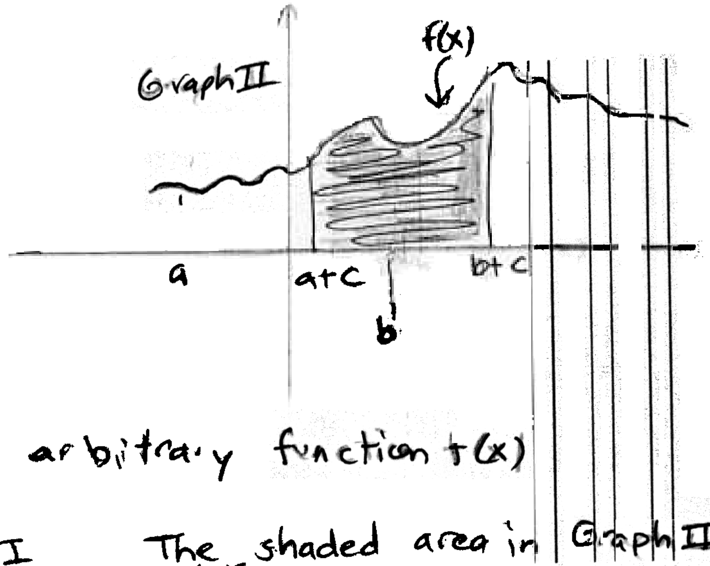
Area $\square = 2(2ab) = 4ab$

By inspection, we find the area of the ellipse, A, can be expressed in the following relation;

$$2ab < A < 4ab.$$ TRUE.

CLAIM 7





I have chosen an arbitrary function f(x) on an x-interval [a,b].

The shaded area in Graph I is $\int_a^b f(x+c)\,dx$ ..

The shaded area in Graph II is $\int_{a+c}^{b+c} f(x)\,dx$

In Graph I, the function f(x) has been shifted to the left c units. (Or, equivalently, the function f(x+c) has been shifted to the right c units. This case is drawn above.).

We can see by inspection that we are finding the area under essentially two identical curves

With the fundamental theorem of Calculus, we would observe the same result,

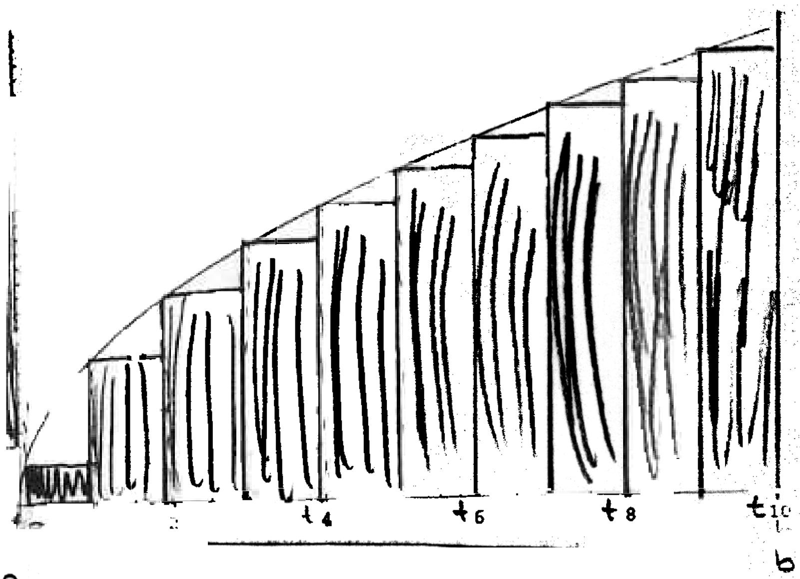
Given $F = f'$

$$\int_a^b f(x+c)\,dx = F(x+c)\Big|_a^b = F(b+c) - F(a+c).$$

$$\int_{a+c}^{b+c} f(x)\,dx = F(x)\Big|_{a+c}^{b+c} = F(b+c) - F(a+c),$$

$\boxed{\text{TRUE.}}$

CLAIM 8 Area under $v(t)$ = total distance



this is not $v(t)=0$

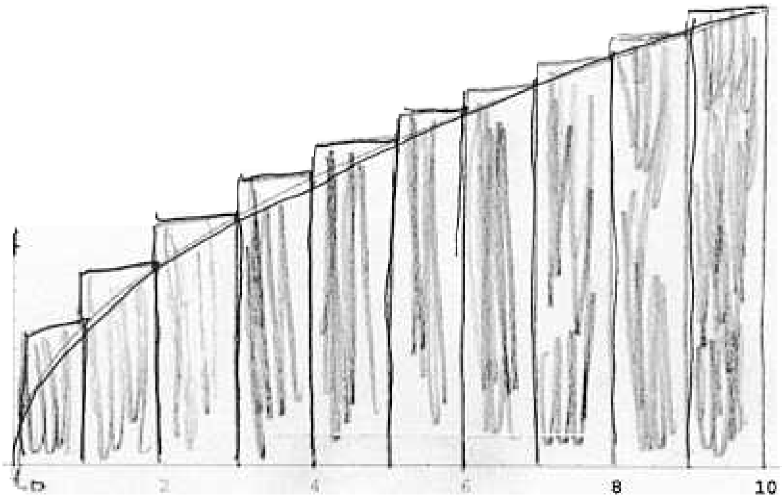
$\sum_{k=1}^{10} v(t_{k-1})\Delta t$ = Approximation of Area under graph using left-hand endpoints.

$\therefore$ This underestimates the area under the curve

$\sum_{k=1}^{10} v(t_{k-1})\Delta t <$ distance traveled on $[a,b]$

CLAIM is FALSE

CLAIM 9



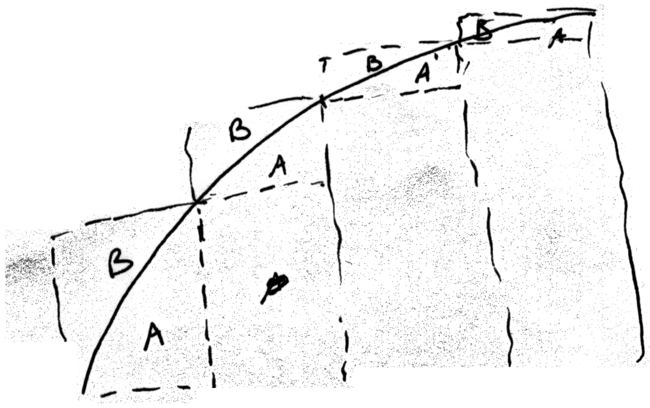
$\sum_{k=1}^{10} v(t_{k-1})\Delta t$ = Approximation of Area under graph using left-hand endpoints

$\therefore$ This overestimates the area under the curve.

$\sum_{k=1}^{10} v(t_k)\Delta t >$ distance traveled on $[a,b]$. $\boxed{\text{CLAIM is TRUE}}$

CLAIM 10

See claim 8 and claim 9 This is averaging the overestimate and the underestimate.



By inspection, we see that A is bigger than B, which means the underestimate is greater than the overestimate. This the average is < the distance traveled on [a, b]

$\boxed{\text{Claim is true}}$

Integration Handout A

5 a) $\int x \cos x \, dx$. Integrate by parts, let $u = x$, $dv = \cos x \, dx$

b) $\int \cos x \sin^2 x \, dx$. Integrate by substitution, let $u = \sin x$

c) $\int \frac{x}{x^2 - 4x - 5} dx$. Integrate by partial fractions.

$$\frac{x}{x^2-4x-5} = \frac{A}{(x+1)} + \frac{B}{(x-5)}.$$

d) $\int \frac{x-2}{x^2-4x+5} dx$. Integrate by substitution, let $u = x^2 - 4x + 5$

$\frac{\ln x}{x} dx$. Integrate by substitution, let $u = \ln x$.

f. $\ln x \, dx$. Integrate by parts, let $u = \ln x$, $dv = dx$.

Integration Handout A - Solutions

6. (a) $\int \frac{x}{\sqrt{4+x^2}}\,dx$ let $u=4+x^2 \Rightarrow du = 2x\,dx$

$=\frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\left[2u^{1/2}\right] = \boxed{\sqrt{4+x^2}+C}$

(b) $\int_0^1 \sqrt{4-t^2}\,dt$ let $t=2\sin\theta \Rightarrow dt=2\cos\theta\,d\theta$

$=\int_0^{\pi/6}\sqrt{4(1-\sin^2\theta)}\cdot 2\cos\theta\,d\theta = 2\int_0^{\pi/6} 2\cos\theta\cdot\cos\theta\,d\theta$

$=4\int_0^{\pi/6}\cos^2\theta\,d\theta = 4\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\Big|_0^{\pi/6}\right] = 4\left[\frac{\pi}{12}+\frac{\sqrt{3}}{8}\right] = \boxed{\frac{\pi}{3}+\frac{\sqrt{3}}{2}}$

(c) $\int_0^1 x^3\sqrt{4-x^2}\,dx$ let $x=2\sin\theta$ $dx=2\cos\theta\,d\theta$

$=\int_0^{\pi/6} 8\sin^3\theta\sqrt{4(1-\sin^2\theta)}\cdot 2\cos\theta\,d\theta = 32\int_0^{\pi/6}\sin^3\theta\underbrace{\cos^2\theta}_{(1-\sin^2\theta)}\,d\theta$

$=32\int_0^{\pi/6}\sin^3\theta\,d\theta - 32\int_0^{\pi/6}\sin^5\theta\,d\theta$

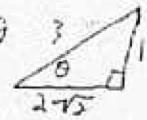
$=32\left[-\frac{2\cos\theta}{3}-\frac{\cos\theta\sin^2\theta}{3}\Big|_0^{\pi/6}\right] - 32\left[-\frac{\cos\theta\sin^4\theta}{5}-\frac{4\cos\theta\sin^2\theta}{15}-\frac{8\cos\theta}{15}\Big|_0^{\pi/6}\right]$

$=\frac{-36\sqrt{3}+64}{3} - \left(\frac{-147\sqrt{3}-256}{15}\right) = \boxed{\frac{64}{15}-\frac{11\sqrt{3}}{5}}$

(d) $\int_0^1 \frac{x^3}{\sqrt{9-x^2}}\,dx$ let $x=3\sin\theta \Rightarrow dx = 3\cos\theta\,d\theta$

$=\int_{x=0}^{x=1}\frac{27\sin^3\theta}{3\cos\theta}\cdot 3\cos\theta\,d\theta$

$=27\int_{x=0}^{x=1}\sin\theta\cdot\sin^2\theta\,d\theta = 27\int_{x=0}^{x=1}\sin\theta(1-\cos^2\theta)\,d\theta$

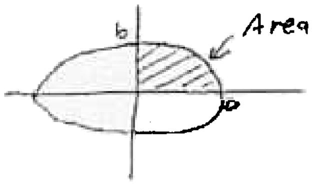


let $u=\cos\theta \Rightarrow du=-\sin\theta\,d\theta$

$=-27\int_{x=0}^{x=1}(1-u^2)\,du = -27\left[\cos\theta-\frac{\cos^3\theta}{3}\Big|_{\theta=0}^{\theta=\arcsin 1/3}\right]$

$=-27\left[\left(\frac{2\sqrt{2}}{3}-\frac{16\sqrt{2}}{81}\right)-\left(1-\frac{1}{3}\right)\right] = \boxed{\frac{-38\sqrt{2}+54}{3}}$

7



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$\Rightarrow A = \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\, dx$$

let $x = a\sin\theta$

$\Rightarrow dx = a\cos\theta\, d\theta$

x	θ
0	0
a	$\pi/2$

$$\Rightarrow A = \int_0^{\pi/2} b\sqrt{1-\sin^2\theta} \cdot a\cos\theta\, d\theta$$

$$= ab\int_0^{\pi/2} \cos^2\theta\, d\theta = ab\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big|_0^{\pi/2}\right]$$

$$= ab\left[\left(\frac{\pi}{4} - 0\right) - (0 - 0)\right] = \frac{ab\pi}{4}$$

$\Rightarrow$ Area of Ellipse $= 4A = \boxed{ab\pi}$