Solutions for Exam 1: March 10, 2004

March 10, 2004

1. **Question A:** (a) and (c)

   For (a), think of cutting vertically: the height of the rectangle is $\pi/2 - \arcsin x$ so we have $\int_0^1 \pi/2 - \arcsin x\, dx = \pi/2 - \int_0^1 \arcsin x\, dx$. Alternatively, think of the area of the rectangle whose height is $\pi/2$ and base is 1 and subtract the area under the arcsine graph from 0 to 1.

   For (c), think of cutting horizontally. the height is Δy and the length is $x = \sin y$.

   Question B: Partition the disk of radius 20 into concentric circular rings. The population in the ith ring is approximately the density times the area of the ring $\approx \rho(x)$(area of ith ring) $\approx \rho(x)(2\pi r_i \Delta x)$. But $r_i = x_i$ so we have

   the population in the ith ring $\approx 100e^{0.4x_i}(2\pi x_i \Delta x)$.

   Total population $= \lim_{n\to\infty} \sum_{i=0}^{n} 100e^{0.4x_i} 2\pi x_i \Delta x) = \int_0^{20} 100e^{0.4x} 2\pi x\, dx = 200\pi \int_0^{20} e^{0.4x} x\, dx$.

2. Cut the oil parallel to the ground, that is, partition the interval $[0, 0.4]$ into n equal pieces, each of length Δh. Each slice of is a rectangle of length 30 m, height Δh m, and width w_i

   The mass of oil in the ith slice $\approx \rho(h_i)30\Delta h w_i$.

   Our job is to find the width in terms of h_i. Draw a circle of radius 0.5 and let $h = 0$ at the bottom of the circle. We can draw a right triangle whose hypotenuse is 0.5, and with legs of $0.5 - h_i$ and x_i.

   $w_i = 2x_i = 2\sqrt{.5^2 - (.5 - h_i)^2}$

   The mass of oil in the ith slice $\approx \rho(h_i)30\Delta h 2\sqrt{.5^2 - (.5 - h_i)^2}$.

   Total mass $= = \lim_{n\to\infty} \sum_{i=0}^{n} \rho(h_i)30\Delta h 2\sqrt{.5^2 - (.5 - h_i)^2} = \lim_{n\to\infty} \sum_{i=0}^{n} \rho(h_i)60\sqrt{.5^2 - (.5 - h_i)^2}\Delta h =$ $60 \int_0^{0.4} \rho(h)\sqrt{.5^2 - (.5 - h)^2}\, dh$.

3. (a) We slice along the y-axis, partitioning the interval $[-4, 0]$ into n equal pieces each of length Δy. The weight for the i-th slice is $\rho(y_i)$ (area of disk at height y_i) $=$ $\rho(y_i)\pi x_i^2 \Delta y$. Expressing x_i in terms of y_i we get

   $$\rho(y_i)\pi(y_i + 4)\Delta y.$$

   So the Riemann sum is

   $$\sum_{i=1}^{n} \rho(y_i)\pi(y_i + 4)\Delta y.$$

(b) Taking the limit of the Riemann sum we get the weight of oil is

$$\int_{-4}^{0} \rho(y)\pi(y+4)dy = \int_{-4}^{0}(10-y)\pi(y+4)dy = \cdots = \frac{688}{3}\pi.$$

(c) Slice the oil along the y-axis. Notice that the distance traveled by the ith slice of oil is $3 - y_i$ We compute work done lifting the slice by multiplying the weight by the distance travelled. The work need to pump the i-th slice up is

$$\rho(y_i)\pi(y_i+4)(3-y_i).$$

Taking the limit of the Riemann sum, we get the total work is

$$\int_{-4}^{0} \rho(y)\pi(y+4)(3-y)dy = \int_{-4}^{0}(10-y)\pi(y+4)(3-y)dy.$$

4. **Solution to Problem 4.**

(a) $\int_1^\infty g(x)\,dx$ could converge or could diverge, so (iii) is true. For instance, $\int_1^\infty \frac{-10}{x}\,dx$ diverges, whereas $\int_1^\infty \frac{-10}{x^2}\,dx$ converges.

(b) (iii) is true as well. For instance, if $h(x) = \frac{1}{x^2}$ then $\sqrt{h(x)} = \frac{1}{x}$ and $\int_1^\infty \frac{1}{x}\,dx$ diverges. If however $h(x) = \frac{1}{x^4}$ then $\sqrt{h(x)} = \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

(c) (i) This is true by comparison, since $f(x) < h(x)$ for $x > 1$ and $\int_1^\infty h(x)\,dx$ converges since h is a p-function with $p > 1$.

(ii) This might possible be true but might also be false. Since $g(x) > h(x)$, we simply do not have enough information about g to decide on the convergence of $\int_1^\infty g(x)\,dx$. For instance, if $g(x) = \frac{1}{x}$ its integral diverges, but if $g(x) = \frac{1}{x^2}$ then its integral converges. Both functions are greater than $\frac{1}{x^3} = h(x)$.

(iii) This is true. $\int_0^3 h(x)\,dx$ diverges because h is a p function with $p = 3 > 1$. Also, $g(x) > h(x)$ for all positive x so $\int_0^3 g(x)\,dx$ diverges as well.

(d) $\int_0^\infty xe^{-x}\,dx$ converges. Its value is in fact 1. Indeed, using integration by parts,

$$\begin{aligned}\int_0^\infty xe^{-x}\,dx &= \lim_{t\to\infty}\int_0^t xe^{-x}\,dx = \lim_{t\to\infty}(-xe^{-x}|_0^t + \int_0^t e^{-x}\,dx) = \lim_{t\to\infty}(-te^{-t} + -e^{-x}|_0^t)\\ &= \lim_{t\to\infty}(-te^{-t} - e^{-t} + 1) = 1.\end{aligned}$$

Note: $\lim_{t\to\infty}(-te^{-t}) = \lim_{t\to\infty}\frac{t}{e^t}$. You can compute this limit using L'Hospital's Rule, since the top and the bottom both grow without bound. $\lim_{t\to\infty}\frac{t}{e^t} = \lim_{t\to\infty}\frac{1}{e^t} = 0$

5. Consider the region S bounded above by the curve $y = \sqrt{x}$, below by $y = \frac{1}{x}$ and on the right by $x = 3$.

(a) Suppose we rotate S about the horizontal line $y = -1$. What is the volume generated?

Solution Partition the region vertically into rectangles with base Δx and height $\sqrt{x} - \frac{1}{x}$. When we rotate about the line $y = -1$ we get washers with volume $(\pi R^2 - \pi r^2)\Delta x$ where $R = (\sqrt{x} + 1)$ and $r = (1/x + 1)$. The volume generated by the ith slice is given by:

$$(\pi(\sqrt{x} + 1)^2 - \pi(1/x + 1)^2)\, \Delta x.$$

$$\begin{aligned} V &= \int_1^3 (\pi(\sqrt{x} + 1)^2 - \pi(1/x + 1)^2) dx \\ &= \pi(2 + 4\sqrt{3} - 2\log 3) \end{aligned}$$

(b) Suppose we rotate S about the vertical line $x = -2$. What is the volume generated?

Solution Partition the region vertically into rectangles with base Δx and height $\sqrt{x} - \frac{1}{x}$. When we rotate about the line line $x = -2$ we get hollow cylindrical shells with volume $2\pi rhDeltax$ where $r = x_i + 2$ and $h = \sqrt{x} - \frac{1}{x}$.
The volume generated by the ith slice is given by:

$$2\pi(x_i + 2)(\sqrt{x_i} - 1/x_i)\, \Delta x.$$

(c)

$$\begin{aligned} V &= \int_1^3 2\pi(x + 2)(\sqrt{x} - 1/x) dx \\ &= 2\pi(\frac{2}{5}\, 3^{\frac{5}{2}} + 4\sqrt{3} - \frac{56}{15} - 2\log 3) \end{aligned}$$

6. Let $f(x) = ke^{-|x|}$, i.e. $f(x) = ke^{-x}$ for $x \geq 0$ and $f(x) = ke^{x}$ for $x < 0$. Suppose f is a probability density function.

(a) What is k? Use this value of k in parts (b), (c), and (d).

(b) What is the median of the distribution?

(c) What is the probability that $0 < X < 2$?

(d) What is the probability that $X > 2$, i.e. $P(X > 2)$?

Solution

(a)

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} ke^{-|x|} dx \\ &= 2k \int_0^{\infty} e^{-x} dx \;=\; 2k \end{aligned}$$

This gives $k = \frac{1}{2}$.

(b) $m = 0$, by symmetry.

(c)

$$P(0 < X < 2) \;=\; \frac{1}{2}\int_0^2 e^{-x}dx \;=\; \frac{1}{2}(1 - e^{-2})$$

(d)

$$P(X > 2) \;=\; \frac{1}{2}\int_2^\infty e^{-x}dx \;=\; \frac{1}{2}e^{-2}$$