Problem Set 14

8.1) 39 8.2) 1, 2, 12, 16, 18, 20

Series Handout 2, 4, 7, 8

2) These are all geometric sums with $r = \frac{1}{3}$

a) $\sum_{k=0}^{100} (1/3)^k$ $\quad a_0 = 1$, $n = 101$, $r = 1/3$ $\rightarrow \sum_{k=0}^{100} \left(\frac{1}{3}\right)^k = \frac{1(1-(1/3)^{101})}{1-1/3}$

$= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{101}\right)$

b) $\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

c) $a_0 = \frac{1}{9}$, $n = 99$, $r = \frac{1}{3}$ $\quad \sum_{k=2}^{100} \left(\frac{1}{3}\right)^k = \frac{\frac{1}{9}\left(1 - \frac{1}{3}^{99}\right)}{1 - 1/3} = \frac{1}{6}\left(1 - (1/3)^{99}\right)$

d) $\sum_{k=2}^{\infty} \left(\frac{1}{3}\right)^k = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \frac{1}{6}$

4) a) Infinite geometric series, so converges if $|x| < 1$ diverges if $|x| \geq 1$ Because if $|x| \geq 1$ $\lim_{n\to\infty} x^n \not\to 0$ so the series could not converge.

for $|x| < 1$ $\quad \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

b) $\sum_{k=0}^{\infty} (x-4)^k$, again geometric series with $r = 4-x$ convergence if $|x-4| < 1$ or $3 < x < 5$, the series diverges if $|x-4| \geq 1$ if $x \geq 5$ or $x \leq 3$

if $|x-4| < 1$ $\quad \sum_{k=0}^{\infty} (x-4)^k = \frac{1}{1-(x-4)} = \frac{1}{5-x}$

7) $f(x) = \sin(\pi x)$ would change her mind. Because for any integer n, $\sin(\pi n) = 0$ so $\lim_{n\to\infty} \sin(\pi n) =$

But for x as real numbers

$\lim_{x\to\infty} \sin(\pi x)$ doesn't exist because the sine function oscillates between -1 and 1.

8) $\sum_{k=1}^{\infty} \frac{1}{2^k+k} = \frac{1}{2+1} + \frac{1}{4+2} + \frac{1}{8+3} + \cdots + a_n + \cdots$

$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + b_n + \cdots$

now notice that the terms of the first series are always smaller

$\frac{1}{2+1} < \frac{1}{2}$, $\frac{1}{4+2} < \frac{1}{4}$, $a_n < b_n$

Then $\sum_{k=1}^{\infty} \frac{1}{2^k+k} < \sum_{k=1}^{\infty} \frac{1}{2^k}$ So by that comparison we would think $\sum_{k=1}^{\infty} \frac{1}{2^k+K}$ converge because $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges.

39. $a_n = \frac{1}{2n+3}$ is decreasing since $a_{n+1} = \frac{1}{2(n+1)+3} = \frac{1}{2n+5} < \frac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$. Note that $a_1 = \frac{1}{5}$.

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is n convergent.

2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5. In other words, it means that $\lim_{n\to\infty} s_n = 5$, where s_n is the nth partial sum, that is, $\sum_{i=1}^{n} a_i$.

12. $1 + 0.4 + 0.16 + 0.064 + \cdots$ is a geometric series with ratio 0.4. The series converges to $\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}$ since $|r| = \frac{2}{5} < 1$.

16. $\sum_{n=1}^{\infty} \left(\frac{1}{e^2}\right)^n \Rightarrow a = \frac{1}{e^2} = |r| < 1$, so the series converges to $\frac{1/e^2}{1-1/e^2} = \frac{1}{e^2-1}$.

18. $\sum_{n=1}^{\infty} \frac{3}{n} = 3\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. [If $\sum_{n=1}^{\infty} \frac{3}{n}$ were to converge, then $\sum_{n=1}^{\infty} \frac{1}{n}$ would also have to converge by Theorem 8(i).] In general, constant multiples of divergent series are divergent.

20. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = \lim_{n\to\infty} \left(1 + \frac{1}{n^2+2n}\right) = 1 \neq 0.$$