Problem Set #15

8.2 : 34, 48, 51 , 8.3 : 3, 4, 9, 10

34. $\displaystyle\sum_{n=0}^{\infty} 2^n(x+1)^n = \sum_{n=0}^{\infty}[2(x+1)]^n = \sum_{n=1}^{\infty}[2(x+1)]^{n-1}$ is a geometric series with $r = 2(x+1)$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|2(x+1)| < 1$ $\Leftrightarrow$ $|x+1| < \frac{1}{2}$ $\Leftrightarrow$ $-\frac{1}{2} < x+1 < \frac{1}{2}$ $\Leftrightarrow$ $-\frac{3}{2} < x < -\frac{1}{2}$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{1}{1-2(x+1)} = \dfrac{1}{-1-2x}$ or $\dfrac{-1}{2x+1}$.

48. If $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim\limits_{n\to\infty}\dfrac{1}{a_n} \neq 0$, and so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{a_n}$ is divergent by the Test for Divergence.

51. The partial sums $\{s_n\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n. Also, the sequence $\{s_n\}$ is bounded since $s_n \le 1000$ for all n. So by Theorem 8.1.7, the sequence of partial sums converges, that is, the series $\sum a_n$ is convergent.

52. (a)
$$\text{RHS} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}} = \frac{f_nf_{n+1} - f_nf_{n-1}}{f_n^2 f_{n-1}f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_nf_{n-1}f_{n+1}} = \frac{(f_{n-1}+f_n) - f_{n-1}}{f_nf_{n-1}f_{n+1}}$$
$$= \frac{1}{f_{n-1}f_{n+1}} = \text{LHS}$$

(b)
$$\sum_{n=2}^{\infty}\frac{1}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}}\right) \quad \text{[from part (a)]}$$
$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1f_2} - \frac{1}{f_2f_3}\right) + \left(\frac{1}{f_2f_3} - \frac{1}{f_3f_4}\right) + \left(\frac{1}{f_3f_4} - \frac{1}{f_4f_5}\right) + \cdots + \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}}\right)\right]$$
$$= \lim_{n\to\infty}\left(\frac{1}{f_1f_2} - \frac{1}{f_nf_{n+1}}\right) = \frac{1}{f_1f_2} - 0 = \frac{1}{1\cdot 1} = 1 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

(c)
$$\sum_{n=2}^{\infty}\frac{f_n}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{f_n}{f_{n-1}f_n} - \frac{f_n}{f_nf_{n+1}}\right) \quad \text{(as above)}$$
$$= \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)$$
$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1} - \frac{1}{f_3}\right) + \left(\frac{1}{f_2} - \frac{1}{f_4}\right) + \left(\frac{1}{f_3} - \frac{1}{f_5}\right) + \left(\frac{1}{f_4} - \frac{1}{f_6}\right) + \cdots + \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)\right]$$
$$= \lim_{n\to\infty}\left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}}\right) = 1 + 1 - 0 - 0 = 2 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

3. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note on page 587.)

(b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

4. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

9. $\dfrac{1}{n^2+n+1} < \dfrac{1}{n^2}$ for all $n \ge 1$, so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+n+1}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

10. $\dfrac{1}{2n-1} > \dfrac{1}{2n} = \dfrac{1}{2}\cdot\dfrac{1}{n}$ for all $n \ge 1$, so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}$ diverges by comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges because it is a nonzero constant multiple of the divergent harmonic series.