Problem Set 18 Solutions

7.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$1+x+x^2$	7
1	$1+2x$	5
2	2	2
3	0	0
4	0	0
$\vdots$	$\vdots$	$\vdots$

$$f(x) = 7 + 5(x-2) + \frac{2}{2!}(x-2)^2 + \sum_{n=3}^{\infty} \frac{0}{n!}(x-2)^n$$
$$= 7 + 5(x-2) + (x-2)^2$$

Since $a_n = 0$ for large n, $R = \infty$.

13.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{4}\right)$
0	$\sin x$	$\sqrt{2}/2$
1	$\cos x$	$\sqrt{2}/2$
2	$-\sin x$	$-\sqrt{2}/2$
3	$-\cos x$	$-\sqrt{2}/2$
4	$\sin x$	$\sqrt{2}/2$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
\sin x &= f\left(\tfrac{\pi}{4}\right) + f'\left(\tfrac{\pi}{4}\right)\left(x-\tfrac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\tfrac{\pi}{4}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{4}\right)}{3!}\left(x-\tfrac{\pi}{4}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{4}\right)}{4!}\left(x-\tfrac{\pi}{4}\right)^4 + \cdots \\
&= \tfrac{\sqrt{2}}{2}\left[1 + \left(x-\tfrac{\pi}{4}\right) - \tfrac{1}{2!}\left(x-\tfrac{\pi}{4}\right)^2 - \tfrac{1}{3!}\left(x-\tfrac{\pi}{4}\right)^3 + \tfrac{1}{4!}\left(x-\tfrac{\pi}{4}\right)^4 + \cdots\right] \\
&= \tfrac{\sqrt{2}}{2}\left[1 - \tfrac{1}{2!}\left(x-\tfrac{\pi}{4}\right)^2 + \tfrac{1}{4!}\left(x-\tfrac{\pi}{4}\right)^4 - \cdots\right] + \tfrac{\sqrt{2}}{2}\left[\left(x-\tfrac{\pi}{4}\right) - \tfrac{1}{3!}\left(x-\tfrac{\pi}{4}\right)^3 + \cdots\right] \\
&= \tfrac{\sqrt{2}}{2}\sum_{n=0}^{\infty}(-1)^n\left[\tfrac{1}{(2n)!}\left(x-\tfrac{\pi}{4}\right)^{2n} + \tfrac{1}{(2n+1)!}\left(x-\tfrac{\pi}{4}\right)^{2n+1}\right]
\end{aligned}$$

The series can also be written in the more elegant form $\sin x = \dfrac{\sqrt{2}}{2}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n(n-1)/2}\left(x-\frac{\pi}{4}\right)^n}{n!}$. If $a_n = \dfrac{(-1)^{n(n-1)/2}\left(x-\frac{\pi}{4}\right)^n}{n!}$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left|x-\frac{\pi}{4}\right|}{n+1} = 0 < 1$ for all x, so $R = \infty$.

18. $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} \Rightarrow f(x) = e^{-x/2} = \sum_{n=0}^{\infty}\frac{(-x/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n\, n!}x^n$, $R = \infty$

20. $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \Rightarrow f(x) = \sin(x^4) = \sum_{n=0}^{\infty}(-1)^n\frac{(x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{8n+4}$, $R = \infty$

22. $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \Rightarrow \cos 2x = \sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\, 2^{2n}}{(2n)!}x^{2n} \Rightarrow$

$f(x) = x\cos 2x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\, 2^{2n}}{(2n)!}x^{2n+1}$, $R = \infty$

Series Handout

(11)

a) $a_0 = +$
$a_1 = +$
$a_2 = -$

b) $b_0 = +$
$b_1 = 0$
$b_2 = -$

c) $c_0 = +$
$c_1 = -$
$c_2 = 0$

d) $d_0 = +$
$d_1 = -$
$d_2 = +$

e) $e_0 = -$
$e_1 = +$
$e_2 = -$

(12) $f(x) = \dfrac{1}{1-x}$ at $x=0$

$$f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \dots + \frac{f^n(0)x^n}{n!}$$

$f(0) = 1$

$$f'(x) = \frac{1}{(1-x)^2} \rightarrow \frac{1}{(1-0)^2} = 1$$

$$f''(x) = \frac{2}{(1-x)^3} \rightarrow \frac{2}{(1-0)^2} = 2$$

$$f^n(x) = \frac{n!}{(1-x)^n} \rightarrow \frac{n!}{(1-0)^n} = n!$$

$$P_n(x) = 1 + x + \frac{2x^2}{2} + \dots + \frac{n!\,x^n}{n!}$$

$$= 1 + x + x^2 + \dots + x^n$$

$$= \sum_{k=0}^{n} x^k$$