Problem Set #2 Math 1B Section 5.6

4. Let $u = \ln x$, $dv = x^4\,dx \Rightarrow du = (1/x)\,dx$, $v = \frac{1}{5}x^5$. Then
$\int x^4 \ln x\,dx = \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^5(1/x)\,dx = \frac{1}{5}x^5 \ln x - \frac{1}{5}\int x^4 dx = \frac{1}{5}x^5 \ln x - \frac{1}{5}\left(\frac{1}{5}x^5\right) + C$
$= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C$, or $\frac{1}{25}x^5(5\ln x - 1) + C$.

6. Let $u = \sin^{-1} x$, $dv = dx \Rightarrow du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$\int \sin^{-1} x\,dx = x\sin^{-1} x - \displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$. Setting $t = 1 - x^2$, we get $dt = -2x\,dx$, so
$-\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}} = -\int t^{-1/2}\left(-\tfrac{1}{2}\,dt\right) = \tfrac{1}{2}\left(2t^{1/2}\right) + C = t^{1/2} + C = \sqrt{1-x^2} + C$. Hence,
$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C$.

9. First let $u = (\ln x)^2$, $dv = dx \Rightarrow du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$. Then by Equation 2,
$I = \int (\ln x)^2\,dx = x(\ln x)^2 - 2\int x\ln x \cdot \frac{1}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \Rightarrow$
$dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x\cdot(1/x)\,dx = x\ln x - \int dx = x\ln x - x + C_1$. Thus,
$I = x(\ln x)^2 - 2(x\ln x - x + C_1) = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

14. First let $u = e^{-\theta}$, $dv = \cos 2\theta\,d\theta \Rightarrow du = -e^{-\theta}\,d\theta$, $v = \frac{1}{2}\sin 2\theta$. Then
$I = \int e^{-\theta}\cos 2\theta\,d\theta = \frac{1}{2}e^{-\theta}\sin 2\theta - \int \frac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right) = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$.
Next let $U = e^{-\theta}$, $dV = \sin 2\theta\,d\theta \Rightarrow dU = -e^{-\theta}\,d\theta$, $V = -\frac{1}{2}\cos 2\theta$, so
$\int e^{-\theta}\sin 2\theta\,d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta\left(-e^{-\theta}\,d\theta\right) = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\,d\theta$. So
$I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I$
$\Rightarrow \frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \Rightarrow$
$I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$.

18. First let $u = x^2 + 1$, $dv = e^{-x}\,dx \Rightarrow du = 2x\,dx$, $v = -e^{-x}$. By (6),
$\int_0^1 (x^2+1)e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx$. Next let
$U = x$, $dV = e^{-x}\,dx \Rightarrow dU = dx$, $V = -e^{-x}$. By (6) again,
$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1$. So
$\int_0^1 (x^2+1)e^{-x}\,dx = -2e^{-1} + 1 + 2(-2e^{-1} + 1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3$.

22. Let $u = \tan^{-1} x$, $dv = x\,dx \Rightarrow du = dx/\left(1+x^2\right)$, $v = \frac{1}{2}x^2$.
Then $\int x\tan^{-1} x\,dx = \frac{1}{2}x^2\tan^{-1} x - \dfrac{1}{2}\displaystyle\int \frac{x^2}{1+x^2}\,dx$. To evaluate the last integral, use long division or observe
that $\displaystyle\int \frac{x^2}{1+x^2}\,dx = \int \frac{(1+x^2)-1}{1+x^2}\,dx = \int 1\,dx - \int \frac{1}{1+x^2}\,dx = x - \tan^{-1} x + C_1$. So
$\int x\tan^{-1} x\,dx = \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x + C_1) = \frac{1}{2}\left(x^2\tan^{-1} x + \tan^{-1} x - x\right) + C$.

28. Let $w = \sqrt{x}$, so that $x = w^2$ and $dx = 2w\,dw$. Thus, $\int_1^4 e^{\sqrt{x}}\,dx = \int_1^2 e^w 2w\,dw$. Now use parts with $u = 2w$,
$dv = e^w\,dw$, $du = 2\,dw$, $v = e^w$ to get $\int_1^2 e^w 2w\,dw = \left[2we^w\right]_1^2 - 2\int_1^2 e^w\,dw = 4e^2 - 2e - 2(e^2 - e) = 2e^2$.

34. (a) Let $u = \cos^{n-1} x$, $dv = \cos x\,dx \Rightarrow du = -(n-1)\cos^{n-2} x\sin x\,dx$, $v = \sin x$ in (2):

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\sin^2 x\,dx\\ &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right)dx\\ &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx\end{aligned}$$

Rearranging terms gives $n\int \cos^n x\,dx = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx$ or

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

(b) Take $n = 2$ in part (a) to get $\int \cos^2 x\,dx = \frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + C$.

(c) $\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\,dx = \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}x + \frac{3}{16}\sin 2x + C$