

Math 1b: Problem Set 21

#20 on Series Handout A

A power series of the form $\sum_{k=0}^{\infty} c_k (x-2)^k$ is centered at $x=2$.

A radius of convergence of 7 leaves four possibilities for the interval of convergence, depending on the endpoints:

- ① $[-5, 9]$, the series converges at both endpoints
- ② $[-5, 9)$, the series converges at $x=-5$ but not at $x=9$
- ③ $(-5, 9]$, the series converges at $x=9$ but not at $x=-5$
- ④ $(-5, 9)$, the series converges at neither endpoint.

#21 on Series Handout A

a) Radius of convergence $R = \frac{1}{2}(7 - (-5)) = \frac{1}{2}(12) = 6.$

$$\left. \begin{array}{l} a - 6 = -5 \\ a + 6 = 7 \end{array} \right\} \boxed{a = 1}$$

b) Since $x = 6.5$ is contained in the interval of convergence,

the series converges for $x = 6.5$.

Since $x = -6.5$ is not contained in the interval of convergence,

the series does not converge for $x = -6.5$.

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If $a_n = \frac{(-1)^n x^n}{n+1}$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{\sqrt{n}}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{\sqrt{n+1}/\sqrt{n}} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{\sqrt{1+1/n}} = |x|$
 By the Ratio Test, the series $\sum \frac{x^n}{n}$ converges when $|x| < 1$, so the radius of convergence $R = 1$. When $x = 1$, the series $\sum \frac{1}{n}$ diverges because it is a p -series with $p = \frac{1}{2} < 1$. When $x = -1$, the series $\sum \frac{(-1)^n}{n}$ converges by the Alternating Series Test. Thus, the interval of convergence $I = [-1, 1)$.

12. If $a_n = n^3(x-5)^n$, $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n} \right| = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^3 |x-5| = |x-5|$. By the Ratio Test, the series converges when $|x-5| < 1 \Leftrightarrow -1 < x-5 < 1 \Leftrightarrow 4 < x < 6$. When $x = 4$, the series becomes $\sum_{n=0}^{\infty} (-1)^n n^3$, which diverges by the Test for Divergence. When $x = 6$, the series becomes $\sum_{n=0}^{\infty} n^3$, which also diverges by the Test for Divergence. Thus, $R = 1$ and $I = (4, 6)$.

14. If $a_n = \frac{(-1)^n x^{2n-1}}{(2n-1)!}$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{x^{2n-1}} \right| = \lim_{n \rightarrow \infty} \frac{x^2}{(2n+1)(2n)} = 0 < 1$ for all x . By the Ratio Test the series converges for all x , so $R = \infty$ and $I = (-\infty, \infty)$.

16. If $a_n = \frac{(-2)^n}{\sqrt{n}}(x+3)^n$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-2)^n(x+3)^n} \right| = \lim_{n \rightarrow \infty} \frac{2|x+3|}{\sqrt{1+1/n}} = 2|x+3| < 1 \Leftrightarrow$$

$$|x+3| < \frac{1}{2} \quad [\text{so } R = \frac{1}{2}] \Leftrightarrow -\frac{7}{2} < x < -\frac{5}{2}. \text{ When } x = -\frac{7}{2}, \text{ the series becomes } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, \text{ which diverges}$$

because it is a p -series with $p = \frac{1}{2} \leq 1$. When $x = -\frac{5}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the

Alternating Series Test. Thus, $I = (-\frac{7}{2}, -\frac{5}{2}]$.

20. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \leq x < 4$ and diverges for at least $x \geq 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$; that is, $\sum c_n (-3)^n$ is convergent.

(d) It diverges when $x = -9$; that is, $\sum c_n (-9)^n = \sum (-1)^n c_n 9^n$ is divergent.