PROBLEM SET #34 12, 13, 16, 17 Supplement
DEH 24cef, 25, 26

24) a) $y''-9y'=0$ $r^2-9r=0$ $r(r-9)=0$
$r=0, 9$ $Ae^{0t}+Be^{9t} = A+Be^{9t}$
$y(x) = A+Be^{9x}$

b) $y''-9y=0$ $r^2-9=0$ $(r-3)(r+3)=0$
$r=-3, 3$ $Ae^{-3t}+Be^{3t}$
$y(x) = Ae^{-3x}+Be^{3x}$

c) $y''+9y=0$ $r^2+9=0$ $r^2=-9$ $r=\pm i$
$y(x)=Ae^{3ix}+Be^{-3ix} = A(i\sin 3x+\cos 3x) + B(i\sin$
$+\cos$

d) $y''-9=0$ → integrate $y'=9t+C_1$
integrate again $y=\frac{9}{2}t^2+C_1t+C_2$

e) $y''-2y'-y=0$ $r^2-2r-1=0$
$r=\frac{2\pm\sqrt{4+4}}{2} = \frac{2\pm 2\sqrt{2}}{2} = 1\pm\sqrt{2}$
$y(x)=Ae^{(1+\sqrt{2})x}+Be^{(1-\sqrt{2})x}$

f) $y''-2y'+2y=0$ $r^2-2r+2=0$
$r=\frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm\sqrt{-4}}{2} = 1\pm i$
$y(x) = Ae^{(1+i)x}+Be^{(1-i)x}$
$=Ae^xe^{ix}+Be^xe^{-ix} = Ae^x(i\sin x+\cos x)$
$e^x(C\sin x+D\cos x)$ $+Be^x(i\sin -x+\cos -$

26) a) want r to be negative for both roots,
example $(x+1)(x+2)$ so
$x^2+3x+2=0$
$x''+3x'+2x=0$ has solution
$4e^{-t}-3e^{-2t}=x(t)$
$\lim_{t\to\infty} x(t)=0$

b) want r to be positive for one of the roots example: $(x-1)(x+1)$
x^2-1
$x''-x=0$ has solution
$-\frac{1}{2}e^{-t}+\frac{3}{2}e^{t}=x(t)$
$\lim_{t\to\infty} x(t)=\infty$

25) a) $x'' = -bx' - cx$ we expect b and c to be positive constants, friction and restoring force.

b) $x'' + bx' + cx = 0$ with $b, c > 0$

roots $\dfrac{-b \pm \sqrt{b^2-4c}}{2}$

Case 1 $b^2 - 4c > 0$ then roots

roots $r_1 = \dfrac{-b - \sqrt{b^2-4c}}{2}$, $r_2 = \dfrac{-b + \sqrt{b^2-4c}}{2}$

Since $c > 0$ $-b + \sqrt{b^2-4c} < 0$ so both roots negative.

solution $Ae^{r_1 t} + Be^{r_2 t} = y(t)$ with $r_1, r_2 < 0$ so $\lim\limits_{t\to\infty} y(t) = 0$

Case 2 $b^2 - 4c = 0$ then one root $-\frac{b}{2}$

solution $Ae^{rt} + Bte^{rt} = y(t)$ with $r = -\frac{b}{2}$

and $\lim\limits_{t\to\infty} y(t) = 0$ to see why

$\lim\limits_{t\to\infty} te^{rt} = 0$ note $\lim\limits_{t\to\infty} \dfrac{t}{\frac{1}{e^{rt}}} = \lim\limits_{t\to\infty} \dfrac{1}{-re^{-rt}}$

$= \lim\limits_{t\to\infty} \frac{1}{-r} e^{rt} = 0$. (use l'hopital

Case 3 $b^2 - 4c < 0$ then imaginary roots

$\dfrac{-b \pm \sqrt{b^2-4c}}{2}$ let $d = \sqrt{b^2-4c}$

Solution $y(t)\ e^{-\frac{b}{2}t}\left[A\cos(dt) + B\sin(dt)\right]$

and $\lim\limits_{t\to\infty} y(t) = 0$ as $\lim\limits_{t\to\infty} e^{-\frac{b}{2}t} = 0$

16) $mx'' = -kx$ $m = 2\,\text{kg}$

$10 = +k \cdot .1$ $k = 100$

then $2x'' + 100x = 0$

general solution is $C_1 \cos 5\sqrt{2}t + C_2 \sin 5$

$x(0) = -2 = C_1 \cos 0 + C_2 \sin 0 = C_1$

$x'(t) = -5\sqrt{2}\, C_1 \sin 5\sqrt{2}t + 5\sqrt{2}\, C_2 \cos 5\sqrt{2}t$

$x'(0) = 0 = -5\sqrt{2}\, C_1 \sin 0 + 5\sqrt{2}\, C_2 \cos 0$

$0 = 5\sqrt{2}\, C_2 \rightarrow C_2 = 0$

Solution is $x(t) = -2 \cos 5\sqrt{2}\, t$

I. $y'' + 4y = 0 \qquad r = \pm 2i$

solution $Ae^{2it} + Be^{-2it} = y(t)$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$y(0) = 2 = A + B$

$y'(0) = 0 = A - B$

$A = 1 \quad B = 1$

then $y(t) = e^{2it} + e^{-2it} = \sum_{n=0}^{\infty} \frac{(2it)^n + (-2it)^n}{n!}$

for n odd $(2it)^n + (-2it)^n = 0$

so for n even $(2it)^n + (-2it)^n = 2(2it)^n$

$= \sum_{n=0}^{\infty} \frac{2(2it)^{2n}}{(2n)!} = 2\left(1 - \frac{2^2t^2}{2!} + \frac{2^4t^4}{4!} \cdots\right.$

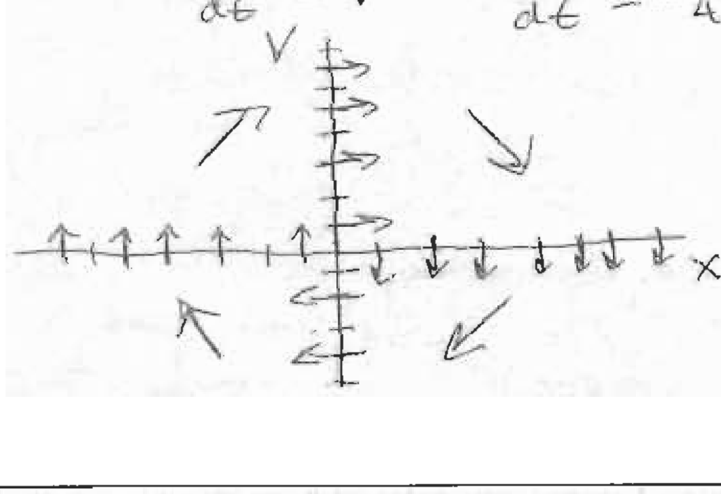
$= 2 \cos(2t)$

II a) $r = \pm\sqrt{-\frac{1}{4}} = \pm \frac{i}{2}$ solution $C_1 \cos \frac{t}{2} + C_2 \sin \frac{t}{2}$

$x(0) = 1 \rightarrow C_1 = 1$ $\quad x'(0) = 0 \rightarrow C_2 = 0$ so

particular solution $\cos \frac{t}{2}$

b) $\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\frac{1}{4}x$

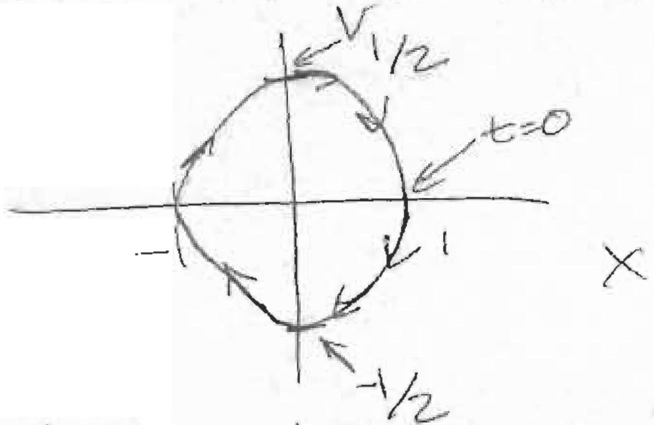


c) $\frac{dv}{dx} = \frac{-\frac{1}{4}x}{v} \rightarrow v\,dv = -\frac{1}{4}x\,dx$

$$\frac{v^2}{2} = -\frac{1}{8}x^2 + C$$

which are ellipses, closed curves, yes this makes sense, since it should oscillate in the frictionless model.

d)



$\frac{dx}{dt} = v = -\frac{1}{2}\sin\frac{t}{2}$

So max of v is $\frac{1}{2}$.

e) $x'' + \frac{1}{10}x' + \frac{1}{4}x = 0$

roots $\frac{-\frac{1}{10} \pm \sqrt{\frac{1}{100} - 1}}{2} = -\frac{1}{20} \pm \frac{\sqrt{\frac{-99}{100}}}{2}$

$= -\frac{1}{20} \pm \frac{i\sqrt{99}}{20}$

$= -\frac{1}{20} \pm \frac{3\sqrt{11}\,i}{20}$

solution is $e^{-\frac{1}{20}t}\left[C_1 \cos\frac{3\sqrt{11}}{20}t + C_2 \sin\frac{3\sqrt{11}}{20}t\right]$

$x(0) = 1 \rightarrow C_1 = 1$

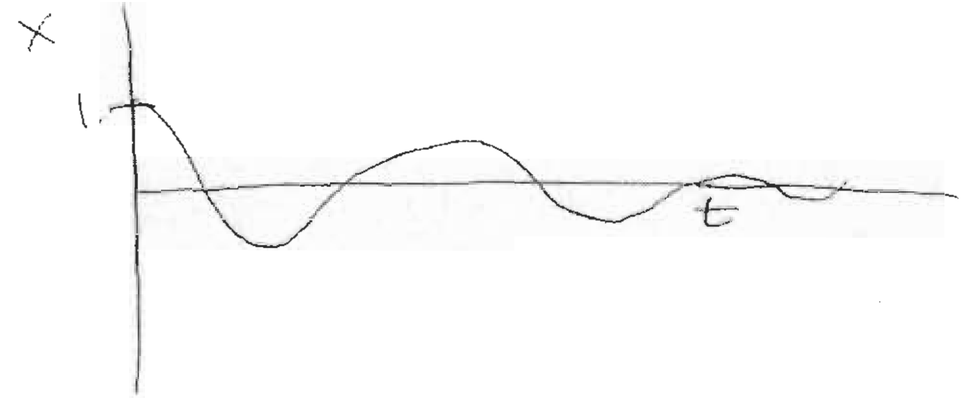
$x'(0) = 0 \rightarrow C_2 = \frac{1}{3\sqrt{11}}$

$x'(t) = e^{-\frac{1}{20}t}\left[-\frac{3\sqrt{11}}{20}C_1 \sin\frac{3\sqrt{11}}{20}t + \frac{3\sqrt{11}}{20}C_2 \cos\frac{3\sqrt{11}}{20}t\right]$

$+ -\frac{1}{20}e^{-\frac{1}{20}t}\left[C_1\cos\frac{3\sqrt{11}}{20}t + C_2 \sin\frac{3\sqrt{11}}{20}t\right]$

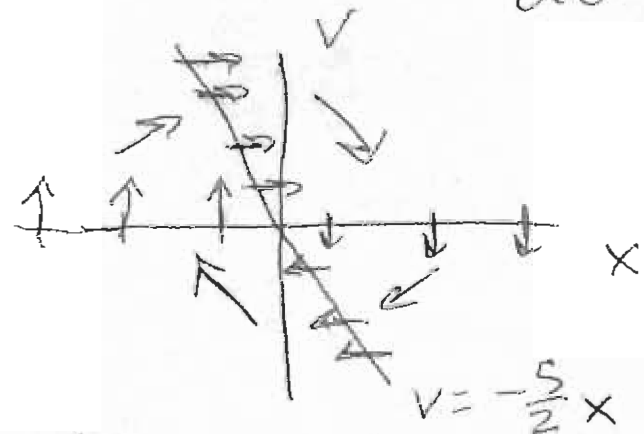
$x'(0) = 0 = \frac{3\sqrt{11}}{20}C_2 + \left(\frac{-1}{20}\right)C_1 \rightarrow C_2 = \frac{1}{3\sqrt{11}}$

$x(t) = e^{-\frac{1}{20}t}\left(\cos\frac{3\sqrt{11}}{20}t + \frac{1}{3\sqrt{11}}\sin\frac{3\sqrt{11}}{20}t\right)$



f) $\frac{dx}{dt} = v$ $\qquad \frac{dv}{dt} = -\frac{1}{10}v - \frac{1}{4}x$

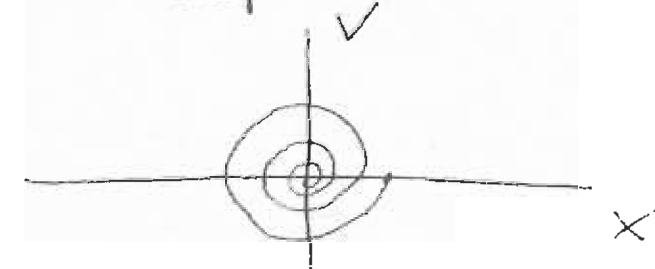
$v = -\frac{5}{2}x$



g) $\frac{dv}{dx} = \frac{-\frac{1}{10}v - \frac{1}{4}x}{v} = -\frac{1}{10} - \frac{1}{4}\frac{x}{v}$

not separable

h)



spiral in

12) In order to have periodic solutions, $x'' + bx' + cx = 0$ must have imaginary roots $a \pm bi$. Then the solution is of the form

$y(x) = e^{ax}[C_3 \cos(bt) + C_4 \sin(bt)]$

13) $x'' + bx' + cx = 0$ $\quad x(n) = 5$ for any integer

so periodic solution with form

$e^{at}[C_3 \cos(bt) + C_4 \sin(bt)]$

$x(0) = 5 \rightarrow 5 = e^0[C_3 \cos 0 + C_4 \sin 0] = C_3$

$x'(t) = e^{at}[-bC_3 \sin(bt) + bC_4 \cos(bt)] + ae^{at}[C_3 \cos bt) + C_4 \sin bt$

$x'(0) = bC_4 + aC_3 = 0 \rightarrow C_4 = -\frac{a}{b} C_3 = \frac{-5a}{b}$

Need period to be 1 so $b = \pm 2\pi$

$5 = e^{an}[5 \cos(\pm 2\pi n) + C_4 \sin(\pm 2\pi n)]$

$5 = 5e^{an} \rightarrow a = 0 \rightarrow C_4 = 0$

solution $\quad 5\cos(\pm 2\pi t)$

roots of polynomial are $\pm 2\pi i$ so

$x'' + (2\pi)^2 x = 0$

17) $e^t \sin t$ then solutions $a + bi = 1 \pm i$

$= \frac{2 \pm 2i}{2}$ which are the solutions to

$r^2 - 2r + 2$ so one solution is

$x'' - 2x' + 2x = 0$