# First Exam for Mathematics 1b

October 27, 2004

| Problem | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 15 | |
| 3 | 16 | |
| 4 | 15 | |
| 5 | 14 | |
| 6 | 15 | |
| 7 | 10 | |
| Total | 100 | |

- You have two hours for this exam.
- Please show all your work on this exam paper.
  Clearly indicate your line of reasoning in order to get full credit.
- NO calculators are permitted.
- Think clearly and do well!

Please circle your section.

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|---|---|---|---|---|---|
| MWF 10 | Erick Matsen | MWF 10 | Rosa Sena-Dias | MWF 11 | Rosa Sena-Dias |
| MWF 11 | Chung Pang Mok | TTH 10 | Rene Reinbacher | TTH 10:00 | Robin Gottlieb |
| TTH 11:30 | Albert Chau | TTH 11:30 | Robin Gottlieb | MWF 12:00 | Weiyang Qiu |

1. (15 points) Your roommate tosses a water balloon filled with mud from your dormitory window into the Quadrangle, making a nice circular splat on the sidewalk below. The density of mud on the sidewalk is $e^{-x}\text{g/cm}^2$, where $x$ is the distance from the center of the mud splat. The splat has a radius of 12 cm.
   (a) Write a general Riemann sum approximating the amount of mud on the sidewalk.
   (b) By taking the appropriate limit of the Riemann sum in (a), arrive at a definite integral giving the amount of mud on the sidewalk.
   (c) Find the amount of mud on the sidewalk.

2. (15 points) To compute the mass of an enormous mountain of height 8500 meters, let us assume it has the shape of a cone with a base radius of 5000. Due to various geological factors such as pressure and heat, the density of the mountain varies with the height $h$ according to the density distribution $\rho(h)$ kg/$m^3$.
   (a) How do you slice the mountain so that the density is approximately constant along each slice.
   (b) Approximate the mass of the ith slice.
   (c) Approximate the total mass by writing a general Riemann sum.
   (d) Write an integral giving the mass of the mountain.

3. (16 points)
   (a) A stand for an old Grecian artifact is going to be constructed as follows. Consider the region R bounded by $y = e^x$, the $x-$axis, the $y-$axis, and the line $x = 1$. Rotate R about the vertical line $x = -1$. Write an integral that gives the volume of the stand. (You need not evaluate.) Please show your work explicitly and clearly so the reader can follow your line of thought. Pictures will be helpful.

(b) A stand for an urn is also going to be constructed. The same region R as in part (a) is now rotated about the line $y = -2$. Write an integral that gives the volume generated. You need not evaluate.) Please show your work explicitly and clearly so the reader can follow your line of thought. Pictures will be helpful.

4. (15 points) A rope is $10m$ long and hangs over the edge of a building. It has a has a $5\,kg$ piece of granite attached to one end. You are required to pull the whole rope over the edge, pulling the granite to the top of the building.

   (a) Suppose the rope has a uniform density of $1kg/m$. Write an integral that gives the amount of work required to pull the rope and the granite up to the top of the building. Feel free to express your answer in terms of g, the gravitational constant. You need not evaluate the integral.

   (b) Suppose that the rope has instead, a non uniform density given by $\rho(x) = 1+x^2$ where $x$ is the distance (in meters) above the granite. Write an integral that gives the amount of work required to pull the rope and the granite up to the top of the building. Feel free to express your answer in terms of g, the gravitational constant. (You need not evaluate the integral.)

   (*Hint: You should slice the rope.*)

5. (14 points) A farmer has a square field with sides of length 1000 m. A water pipe lies along one diagonal of the field. Position the field so that the water pipe lies along the $x$-axis and the other diagonal of the field lies along the $y$-axis . Assume that the harvested yield per area depends only on the distance $d$ from the water pipe and is given by $H(d)$ kg/$m^2$. . We want to determine the total harvest, in kilograms.

   (a) Give a good slicing of the field. Indicate this by a sketch and in words.

   (b) How many kilograms are harvested from the whole field?

6. (15 points)

   (a) Find $\int_e^T \frac{1}{x \ln x}\,dx$.

   (b) Does the integral $\int_e^\infty \frac{1}{x \ln x}\,dx$ converge or diverge? If it converges, what does it converge to?

   (c) Does $\int_e^\infty \frac{1}{x}\,dx$ converge or diverge?
   We know that $\frac{1}{x} \geq \frac{1}{x \ln x}$ for $x \geq e$. Would this comparison alone allow us to determine whether or not $\int_e^\infty \frac{1}{x \ln x}\,dx$ converges? Why or why not?

   (d) Consider the integral $\displaystyle\int_e^\infty \sqrt{\frac{1}{x \ln x}}\,dx$. Can you determine whether or not it converges without finding an antiderivative of $\sqrt{\frac{1}{x \ln x}}$? If so, what do you conclude and what is your rationale? Explain your reasoning carefully and clearly.

7. (10 points) For which value of $c$ does the line $y = c$ separate the region bounded by the curves $y = x^2$ and $y = 9$ into two regions of equal area?

   (Note: if the algebra is intractable for the method you've chosen, you might try an alternate slicing method.)