Solutions to the First Examination: Mathematics1b
March 9, 2005

1. Slice the area in question vertically (parallel to the $y-$ axis). In other words, partition the interval $[0, \pi]$ into n equal pieces, each of length Δx.

   The ith slice generates a cylindrical shell when rotated about the line $x = 3.5$.

   The volume of a cylindrical shell $\approx 2\pi r h \Delta x$

$$r = 3.5 - x$$
$$h = 0 - y = 0 - (-4\sin x) = 4\sin x.$$

   So the volume of the ith slice is $\approx 2\pi x_i \, 4\sin x_i \, \Delta x$

$$\text{Answer: } \int_0^\pi 2\pi \, x \, 4\sin x \, dx = 8\pi \int_0^\pi \sin x \, dx$$

2. (11 points) Short Answer:

   (A.) Slicing vertically gives (a) $\int_0^1 (3\pi/2 - 3\arcsin x) \, dx$

   Slicing horizontally gives (c) $\int_0^{3\pi/2} \sin(\frac{y}{3}) \, dy$

   (B) We know f is defined and continuous on $[10, \infty)$ and $\displaystyle\int_{10}^\infty f(x) \, dx$ diverges, but $\displaystyle\lim_{b\to\infty} \left| \int_{10}^b f(x) \, dx \right| \neq \infty$.

   Choose a continuous function f such that $\lim_{b\to\infty} \int_{10}^b f(x) \, dx$ does not exist. For example, $f(x)$ could be $\sin x$ or $\cos x$.

3. First we figure out the volume used, then we divide that by the total possible volume and multiply by 100 to get a percentage.

   Let's call the height of the water h. Most students either took h running from 0 to b or from -3 to $b - 3$. Both methods can work.

   Let's take them in turn.

   (a) $0 \le h \le b$

   At height h_i a careful application of the Pythagorean theorem tells us that the radius of the i^{th} slice is $\sqrt{(9 - (3 - h_i)^2)}$ so the volume of the i^{th} slice is $\pi(9 - (3 - h_i)^2)\Delta h$. This gives the final integral as

$$\int_0^b \pi(9 - (3 - h)^2) \, dh = \pi(3b^2 - b^3/3)$$

   (b) $-3 \le h \le b - 3$

   At height h_i a careful application of the Pythagorean theorem tells us that the radius of the i^{th} slice is $\sqrt{(9 - h_i^2)}$ so the volume of the i^{th} slice is $\pi(9 - h_i^2)\Delta h$. This gives the final integral as

$$\int_{-3}^{b-3} \pi(9 - h^2) \, dh = \pi(3b^2 - b^3/3)$$

   Either method yields the same answer. We now have to divide by $\frac{4\pi 3^3}{3}$ and multiply by 100 to get our final answer:

$$\frac{100(3b^2 - b^3/3)}{36}$$

4. Eliza and Awinja both do the same amount of work lifting just the buckets from the bottom of their respective wells, namely $4.7.g$Joules

   However the ropes they use both have the same mass-density but Awinja has more rope to pull up! Hence it's pretty clear that Awinja does more work in total.

   Let's figure out how much work each does on their rope and then take the difference.

   There are two ways to proceed - slicing the rope or slicing the distance

   (a) Slicing the rope.

   Let's slice Awinja's rope up (poor Awinja!) and figure out how much work she does on each little piece of the rope. Call the top of the rope $h = 0$ and the bottom of the rope $h = 7$. Now, our i^{th} slice has to travel distance h_i to get to the top of the well, and our i^{th} slice (like all our slices) has mass $0.5\Delta h$ kilograms. Hence the work done to get the i^{th} slice to the top of the well is

   $$h_i 0.5g\Delta h$$

   So the total work done is

   $$\int_0^7 0.5gh\, dh$$

   Let's slice Eliza's rope up and figure out how much work she does on each little piece of the rope. Call the top of the rope $h = 0$ and the bottom of the rope $h = 4$. Now, our i^{th} slice has to travel distance h_i to get to the top of the well, and our i^{th} slice (like all our slices) has mass $0.5\Delta h$ kilograms. Hence the work done to get the i^{th} slice to the top of the well is

   $$h_i 0.5g\Delta h$$

   So the total work done is

   $$\int_0^4 0.5gh\, dh$$

   When you evaluate these and take the difference you find that Awinja does $(12.25 - 4)g = 8.25g$ Joules of work more than Eliza.

   (b) Slicing the distance.

   The distance h in Awinja's case is between 0 and 7. The work done in pulling up the rope from distance h_i to h_{i+1} is

   $$(7 - h_i)0.5g\Delta h$$

   because the length of rope hanging down has length $7 - h_i$ and hence weight $(7 - h_i)0.5g$ and the distance it moves from h_i to h_{i+1} is Δh.

   Hence the total work done in lifting the whole rope to the surface is

   $$\int_0^7 (7 - h)0.5h\, dh$$

   The distance h in Eliza's case is between 0 and 4. The work done in pulling up the rope from distance h_i to h_{i+1} is

   $$(4 - h_i)0.5g\Delta h$$

   because the length of rope hanging down has length $4 - h_i$ and hence weight $(4 - h_i)0.5g$ and the distance it moves from h_i to h_{i+1} is Δh.

   Hence the total work done in lifting the whole rope to the surface is

$$\int_0^4 (4-h)0.5h\,dh$$

When you evaluate these and take the difference you find that Awinja does
$(12.25-4)g = 8.25g$ Joules of work more than Eliza.

5. (a) We choose our axis of integration to be the y-axis, which is vertical, starting with zero at surface level and pointing downwards. We slice the volume in vertical slices since the density is approximately constant along them. In oder to compute work which is necessary to lift the slice sand at the point y_i we use

$$W(y_i) = F(y_i)y_i$$

Here $F(y_i)$ denotes the force of gravity on the slice, and y_i the distance, which our slice has to travel to the surface. Since

$$F(y_i) = m(y_i)g = \rho(y_i)V(y_i)g$$

and the density $\rho(y_i)$ is given, we only need to compute the volume $V(y_i)$ of the slice at y_i. By comparing similar triangles we find

$$\frac{r(y_i)}{0.5-y_i} = \frac{0.2}{0.5}$$

where $r(y_i)$ denotes the radius of the slice at y_i. Using the formula

$$V(y_i) = r(y_i)^2\pi\Delta y$$

for the volume of the slice at y_i, and, after summing over all slices we find for the total work

$$W = \int_0^{0.5} 900(1+\frac{y}{y+1})(\frac{2}{5}(0.5-y))^2\pi g dy.$$

(b) The regions of uniform density are *horizontal* disks, so we slice in this direction. A slice of height Δy will have mass

$$\Delta m \approx \rho(y)\Delta V(y) = \rho(y)\pi r(y)^2\Delta y.$$

In this case, $r = x$, and if $y = -x^3+1$, then $x = (1-y)^{1/3}$. Thus

$$\Delta m \approx \pi\rho(y)(1-y)^{2/3}\Delta y.$$

Forming a Riemann sum we have

$$m \approx \sum_{i=1}^{n} \rho(y_i^*)(1-y_i^*)^{2/3}\Delta y,$$

or, in passing to the integral

$$m = \int_0^1 \rho(y)(1-y)^{2/3}\,dy.$$

(c) i. This time, the regions where the density of ants is roughly constant are circles around the crumb of bread. If ΔN is the number of ants in an "annulus" (slightly thickened circle) around the center with radius x and thickness Δx, then

$$\Delta N \approx \rho(x)\Delta A(x) \approx (10-\ln(1+x^2))2\pi x\,\Delta x.$$

Thus a Riemann sum approximating the number of ants within a 20cm radius is

$$N \approx \sum_{i=1}^{n} 2\pi x_i^* \left(10-\ln(x_i^{*2}+1)\right)\Delta x,$$

where $\Delta x = \frac{20}{n}$, and x_i^* is some point between x_{i-1} and x_i.

ii. As $n->\infty$, the Riemann sum converges to

$$N = 2\pi \int_0^{20} x\left(10 - \ln(1+x)^2\right)\, dx.$$

iii. There are a couple of ways to do this integral. Perhaps the most obvious (but not quickest) is to split it up and integrate by parts:

$$\begin{aligned} N &= 2\pi \int_0^{20} 10x\, dx - 2\pi \int_0^{20} \underbrace{\ln(x^2+1)}_{u}\, \underbrace{x\, dx}_{dv} \\ &= 2\pi(5x^2)\Big|_0^{20} - 2\pi \left\{ \frac{1}{2}x^2 \ln(x^2+1)\Big|_0^{20} - \int_0^{20} \frac{x^3}{x^2+1}\, dx \right\} \end{aligned}$$

With long division, we find that

$$\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1},$$

so

$$\begin{aligned} N &= 4000\pi - 400\pi \ln(401) - 2\pi \left(\frac{x^2}{2}\right)\Big|_0^{20} - \pi \ln(x^2+1)\big|_0^{20} \\ &= 4400\pi - 401\pi \ln(401). \end{aligned}$$

However, another way is through greedy substitution $u = x^2+1$. Then $du = 2x\, dx$ and the integral becomes

$$\begin{aligned} N &= \pi \int_0^{401} (10 - \ln u)\, du \\ &= \pi \left[10u - (u \ln u - u)\right]_0^{401} \\ &= \pi \left[11u - u \ln u\right]_1^{401} \\ &= 4400\pi - 401 \ln(401). \end{aligned}$$

6. (a) Direct computation gives

$$\int_0^\infty \frac{1}{1+x^2} dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2} dx = \lim_{t\to\infty} \arctan|_0^t = \frac{\pi}{2}.$$

(b) First we observe that the integrand obeys

$$0 < \frac{\sin x + 2}{1+x^2} < \frac{4}{1+x^2}$$

for positive x. In addition, we know from above that

$$\int_1^\infty \frac{1}{1+x^2} dx$$

converges, hence

$$4\int_1^\infty \frac{1}{1+x^2} dx$$

converges and

$$\int_1^\infty \frac{\sin x + 2}{1+x^2} dx$$

converges by comparison.

(c) Consider

$$\int_0^\infty \frac{1}{1+x^5}dx = \int_0^1 \frac{1}{1+x^5}dx + \int_1^\infty \frac{1}{1+x^5}dx.$$

The first term is a proper integral, hence it converges. For the second term we use the estimate

$$\frac{1}{1+x^5} \leq \frac{1}{x^5}$$

for positive x and the fact that

$$\int_1^\infty \frac{1}{x^5}dx$$

converges. Hence the second term converges by comparison and the total integral converges as well.

(d) The integral has two sources of improperness. The discontinuity of the integrand at $x = 1$ and the upper boundary at infinity. Hence we split the integral as follows

$$\int_0^\infty \frac{1}{(x-1)^2}dx = \int_0^1 \frac{1}{(x-1)^2}dx + \int_1^2 \frac{1}{(x-1)^2}dx + \int_2^\infty \frac{1}{(x-1)^2}dx$$

We will show that the first term diverges. After a we substitute $u = x - 1$ we find

$$\int_0^1 \frac{1}{(x-1)^2}dx \int_{-1}^0 \frac{1}{u^2}du$$

which clearly diverges. Hence our integral is divergent.