## Answers to the Second Math 1b Exam: November 22, 2004

1. Part (a) From the definition of Taylor polynomial in the first part we have $c_0 = f(0)$, $c_1 = f^{'}(0)$ and $c_2 = \frac{f^{''}(0)}{2}$. So we need only determine the signs of $f(0)$, $f^{'}(0)$ and $f^{''}(0)$. Try drawing a picture of $f$. This will be the simplest way to answer these questions. If you have trouble, note the following.

   Since $f$ is odd this means $f(x) = -f(-x)$. Substitute in $x$ by zero in the previous equation and you get $f(0) = -f(0)$ so $f(0)$ is zero.

   (By taking derivatives on both sides of $f(x) = -f(-x)$ one gets $f'(x) = f^{'}(-x)$ (so $f^{'}$ is even) and by doing it again one gets $f''(x) = -f^{''}(-x)$ so that $f^{''}$ is odd and $f''(0) = 0$.)

   For the sign of $f^{'}(0)$, one needs to note that it is said that $f$ is decreasing on $(0, 3)$ therefore $f^{'}(0) < 0$.

   If $f$ is concave up on (0, 6) then it is concave down on (-6, 0) and therefore, since f is twice differentiable, at 0 we know $f^{''}$ is 0.

   Part (b) For the second part again from the definition of Taylor polynomial we have $c_0 = f(3)$, $c_1 = f^{'}(3)$ and $c_2 = \frac{f^{''}(3)}{2}$. So we need only determine the signs of $f(3)$, $f^{'}(3)$ and $f^{''}(3)$. $f$ is decreasing on $(0, 3)$ therefore $0 = f(0) > f(3)$. But $f$ starts increasing at 3 therefore 3 has to be a local minimum and $f^{'}(3) = 0$. The concavity of $f$ allows us to conclude that $f^{''}(3) > 0$.

2. (a) The best integer approximation of $26^{\frac{1}{3}}$ is clearly 3 with $3^3 = 27$.

   (b) To write the Taylor polynomial to fourth degree first find the various derivatives of $f(x) = x^{\frac{1}{3}}$. We find

   $$f^{(1)}(x) = \frac{1}{3}x^{-\frac{2}{3}}, \quad f^{(2)}(x) = -\frac{2}{3^3}x^{-\frac{5}{3}}, \quad f^{(3)}(x) = \frac{10}{3^3}x^{-\frac{8}{3}}, \quad f^{(4)}(x) = -\frac{80}{3^4}x^{-\frac{11}{3}} \tag{1}$$

   .

   We will expand our Taylor polynomial around $a = 27$. This is the correct choice, since 27 is the nearest point to 26 where we can compute the coefficients in the Taylor polynomial without any additional help. We find for the fourth degree Taylor polynomial of $f(x)$

   $$f(27) + \frac{f'(27)}{1}(x-27) + \frac{f''(27)}{2!}(x-27)^2 + \frac{f'''(27)}{3!}(x-27)^3 + \frac{f''''(27)}{4!}(x-27)^4 \tag{2}$$

   $$3 + \frac{1}{3^3}(x-27) - \frac{1}{3^7}(x-27)^2 + \frac{5}{3^{12}}(x-27)^3 - \frac{10}{3^{16}}(x-27)^4 \tag{3}$$

   (c) To approximate $26^{\frac{1}{3}}$ we evaluate our Taylor polynomial at $x = 26$ and find

   $$3 - \frac{1}{3^3} - \frac{1}{3^7} - \frac{5}{3^{12}} - \frac{10}{3^{16}}. \tag{4}$$

   (d) A closer inspection of the above Taylor ploynomial shows that all higher order terms will be negative. Since these terms will improve our result, the result above is to large.

   (e) The terms are getting smaller. A general rule of thumb is that the magnitude of the error is about the same order of magnitude as the first unused term. $\frac{10}{3^{16}}$ should be an upper bound for the error. (No points were deleted for this part of the question.)

3. (a) Since the Taylor expansion for $e^x$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ then we can substitute to get the Taylor expansion for $e^{-t^2}$:

   $$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot t^{2n}}{n!} = 1 - t^2 + \frac{t^4}{2} + \ldots$$

   (b) We can integrate the above expression to get

   $$\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot t^{2n+1}}{(2n+1) \cdot n!}$$

   therefore

   $$\int_0^{0.1} e^{-t^2}\,dt \approx 10^{-1} - \frac{1}{3}10^{-3} + \frac{1}{10}10^{-5} - \frac{1}{42}10^{-7}$$

By the alternating series error estimate, the error is less than the first ignored term. In this case $10^{-9}/(9 \cdot 4!) < 10^{-10}$ and so we know that the error is within the required bound. Full credit was given for expanding out farther, however it was necessary to state that the series was alternating to get full credit.

(c) The series is alternating and decreasing in absolute value. Therefore if your last term was negative, then your approximation was too small, and if it was positive, your approximation was too big.

4. (a). The ball will travel

$$50 + 2(50 \cdot (0.6) + 50 \cdot (0.6)^2).$$

Notice that after the ball hits the ground for the first time, it will travel up, then travel down before it hits the ground the second time.

(b). A similar consideration as that in part(a) gives the distance

$$D = 50 + 2\sum_{n=1}^{34} 50(0.6)^n.$$

We need to find $\sum_{n=1}^{34} 50(0.6)^n$.

Let $S = 50 \cdot (.6) + 50 \cdot (.6)^2 + \ldots + 50 \cdot (.6)^{34}$

$.6S = \quad 50 \cdot (.6)^2 + \ldots + 50 \cdot (.6)^{34} + 50 \cdot (.6)^{35}$ Subtracting gives

$.4S = 50 \cdot (.6) - 50 \cdot (.6)^{35}$

So

$$S = \frac{50 \cdot (.6) - 50 \cdot (.6)^{35}}{.4} = 2.5(50 \cdot (.6) - 50 \cdot (.6)^{35})$$

$$\sum_{n=1}^{34} 50 \cdot (0.6)^n = 2.5(50 \cdot (.6) - 50 \cdot (.6)^{35}).$$

Therefore, the distance

$$D = 50 + 2[2.5(50 \cdot (.6) - 50 \cdot (.6)^{35})] = 50 + 250((.6) - (.6)^{35})$$

5. Let's apply the ratio test to

$$\sum_{n=2}^{\infty} \frac{(-1)^n (x+5)^n}{2^n \ln(n)} \tag{5}$$

We have:

$$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{\frac{(-1)^{n+1}(x+5)^{n+1}}{2^{n+1}\ln(n+1)}}{\frac{(-1)^n (x+5)^n}{2^n \ln(n)}} \right| = \frac{|(x+5)| \ln(n+1)}{2\ln(n)} \tag{6}$$

To calculate $\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)}$, since both numerator and denominator tend to $\infty$, we can change to $x$ and apply L'hopital's rule to get

$$\lim_{x\to\infty} \frac{\ln(x+1)}{\ln(x)} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} = 1. \tag{7}$$

Therefore

$$\lim_{n\to\infty} \left|\frac{(x+5)\ln(n+1)}{2\ln(n)}\right| = \lim_{n\to\infty} \frac{|x+5|}{2} \tag{8}$$

and we conclude that the series converges when $\frac{|x+5|}{2} < 1$ and diverges for $\frac{|x+5|}{2} > 1$. Sorting out the inequality we get that the series converges on $(-7, -3)$. We need to treat the points $-7, -3$ separately since the ratio test is inconclusive there. At $-7$ the series is

$$\sum_{n=2}^{\infty} \frac{(-1)^n(-7+5)^n}{2^n \ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^{2n} 2^n}{2^n \ln(n)} = \sum_{n=2}^{\infty} \frac{1}{\ln(n)} \tag{9}$$

Since $\ln(n) < n$ for $n > 2$ we know $\frac{1}{\ln(n)} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent so is $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ (direct comparison for series with positive general term). At $-3$ the series is

$$\sum_{n=2}^{\infty} \frac{(-1)^n(-3+5)^n}{2^n \ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{2^n \ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}. \tag{10}$$

This is an alternating series whose general terms is decreasing and converges to zero so by the alternating series test it is convergent. The interval of convergence of the series is therefore $(-7, -3]$.

6. (a) The power series is centered at $x = 3$. Since it converges at $x = -2$, we know that the radius of convergence must be at least $R = |3 - (-2)| = 5$. Since it diverges at $x = -3$, we know that the radius of convergence can be at most $R = |3 - (-3)| = 6$. Thus we can say that the interval of convergence is at least $[-2, 8)$ and that the series converges absolutely for $x \in (-2, 8)$. The interval of convergence is no larger than $(-3, 9]$.

   - $|3 - 2| = 1 < 5$ and so $x = 2$ is well within the minimal radius of convergence from 3, and so the series must converge here.
   - $|3 - 8| = 5$ and so $x = 8$ is a potential end point of the interval of convergence, and so the series may or may not converge here.
   - $|3 - 9| = 6$ and so $x = 9$ is a potential end point of the interval of convergence, and so the series may or may not converge here.
   - $|3 - 10| = 7$ and so $x = 10$ is well outside the maximal radius of convergence from 3, and so the series must diverge here.

   (b) We know that the series converges for $x = 4$ and thus, $\sum_{n=0}^{\infty} c_n(4-3)^n = \sum_{n=0}^{\infty} c_n$ converges. In fact, we know the series converges absolutely, at $x = 4$, so

$$\sum_{n=0}^{\infty} |c_n|$$

   converges. Thus by the Nth term test, we know that $\lim_{n\to\infty} |c_n| = 0$, and in particular we know that for $n$ sufficiently large, $|c_n| < 1$. From this we may conclude that for $n$ sufficiently large, $c_n^2 < |c_n|$ and thus $\sum_{n=0}^{\infty} c_n^2$ converges by comparing to $\sum_{n=0}^{\infty} |c_n|$..

   If for some reason you don't want to use absolute convergence in your argument, you can show the convergence as follows:

   We know that the series converges for $x = 5$ and thus, $\sum_{n=0}^{\infty} c_n(5-3)^n = \sum_{n=0}^{\infty} c_n 2^n$ converges. Thus by the divergence test, we know that $\lim |c_n|2^n = 0$, and in particular we know that for $n$ sufficiently large, $|c_n|2^n < 1$. From this we may conclude that for $n$ sufficiently large, $|c_n| < \frac{1}{2^n}$, and thus $c_n^2 < \frac{1}{4^n}$, and thus $\sum_{n=0}^{\infty} c_n^2$ converges by comparing to a geometric series.

7. Part(i) Use the fact that $|\frac{e^{-n}cos(n)}{n!}| \leq e^{-n}$ for any non-negative integer n. Now the geometric series $\sum_{n=1}^{\infty} e^{-n}$ converges because it is a geometric series with ration of absolute value strictly smaller than one. By comparison test, $\sum_{n=1}^{\infty} |\frac{e^{-n}cos(n)}{n!}|$ converges, i.e. $\sum_{n=1}^{\infty} \frac{e^{-n}cos(n)}{n!}$ is absolutely convergent

Part(ii) Notice that $\lim_{n\to\infty} \frac{n^2}{5n^2+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{5+n^{-3/2}}$ =1/5. As a result, $\lim_{n\to\infty}(-1)^n \frac{n^2}{5n^2+\sqrt{n}}$ does not exist. In particular, $\lim_{n\to\infty}(-1)^n \frac{n^2}{5n^2+\sqrt{n}}$ is not zero. So by n - th term test for divergence, $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{5n^2+\sqrt{n}}$ diverges

Part(iii) Let $a_n = \frac{10n}{n^3-n+1}$, $b_n = \frac{1}{n^2}$. Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{10n^3}{n^3-n+1} = \lim_{n\to\infty} \frac{10}{1-n^{-2}+n^{-3}} = 10$, which is a nonzero finite number. Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (p-series with p=2), we see, by the limit comparison test, that $\sum_{n=2}^{\infty} \frac{10n}{n^3-n+1}$ converges. Since the summands are positive, the series is absolutely convergent.

Part (iv) Let $b_n = \frac{1}{n \ln n}$. Then $b_n$ form a monotomic decreasing sequence of nonnegative numbers, and $\lim_{n\to\infty} b_n$ =0 ,so by alternating series test, $\sum_{n=2}^{\infty}(-1)^n b_n = \sum_{n=2}^{\infty}(-1)^n \frac{1}{n \ln n}$ converges.

We show that it is not absolute convergent. Indeed To show that $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges, it suffices, by the integral test, to show that $\int_2^{\infty} \frac{1}{x \ln x}\, dx$ diverges. But $\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty}(\ln \ln t - \ln\ ln 2) = \infty$ , (using the substitution $u =\ lnx$) i.e. the improper integral diverges. Thus the series is only conditionally convergent

8. 1. (a) False

   We know the series $\sum_{k=1}^{\infty} a_k$ converges, so the terms of the series must go to zero, not $\pi$.

   (b) True.

   We know $\lim_{k\to\infty} a_k = 0$; otherwise the series would diverge by the Nth Term Test for Divergence.

   (c) True.

   This is exactly what the statement $\sum_{k=1}^{\infty} a_k = \pi$ means.

   (d) May be true or may be false.

   If all the terms are positive then it would certainly be true. But on the other hand, $s_n$ could approach $\pi$ from above. For instance, if all the terms after the first one are negative then it would be false. Alternatively, $s_n$ could sometimes be more than $\pi$ and sometimes less.

   (e) False.

   By the ratio test we know that this limit can't be more than one.

   (f) May be true or may be false.

   We only know that if this limit exists that it is less than or equal to one.

   (g) May be true or may be false.

   We only know that if this limit exists that it is less than or equal to one. (If the limit is 1 then the ratio test is inconclusive, but that doesn't mean that the series doesn't converge.)

2. (a) True.

   If $f$ is a polynomial then the series will be finite. A polynomial has only finitely many non-zero derivatives.

   (b) False

   The Taylor series for $\frac{1}{1-x}$ is $\sum_{k=1}^{\infty} x^k$. This series converges only for $|x| < 1$, so the statement cannot be true.

   (It is possible that the series converges for all $x$, but it is possible that it does not.)

   (c) True.

   The terms are all negative, so the partial sums form a decreasing sequence. It is bounded above by 0 and below by -100. Therefore, by the Monotonic Bounded Sequence Theorem, the sequence of partial sums, and hence the series, converges.