Problem Set #30

Differential Equations Handout

13. a) Competitive, because the xy terms have negative coefficients in both cases.

b) $.05xy = .1x \Leftrightarrow y = 2$
$.05xy = .1y \Leftrightarrow x = 2$

c) See page 3

d) In the quadrant $0 < x < 2, 0 < y < 2$, the general direction of the trajectories is positive for both coordinates.

$2 < x, 0 < y < 2$ negative in y, positive in x

$2 < x, 2 < y$ negative in y, negative in x

$0 < x < 2, 2 < y$ positive in y, negative in x

e) If $x = 0$, y increases without bound. At $x = 0$, i.e. y-axis arrows point vertically up.
If $y = 0$, x increases without bound. At $y = 0$, i.e. x-axis arrows point horizontally right.

f) See page 3

g) i. Beast X increases and Beast Y starts steady and then decreases.
ii. Beast X decreases and Beast Y starts steady and then increases.
iii. Beast X starts steady and then increases, Beast Y decreases.

h) Supports Darwin because in the long run, one of the species wins.

15. See page 4 and 5.

Stewart

7.6

1 a) x = predator with no additional food sources.
y = prey with no additional growth restrictions.

b) x = prey restricted by predators and carrying capacity.
y = predator with no additional food sources.

2 a) Cooperative model b) Competitive model

Chapter 7
Review

20. a) y = bird, x = insects. Many more insects; insects are prey, birds are predators.

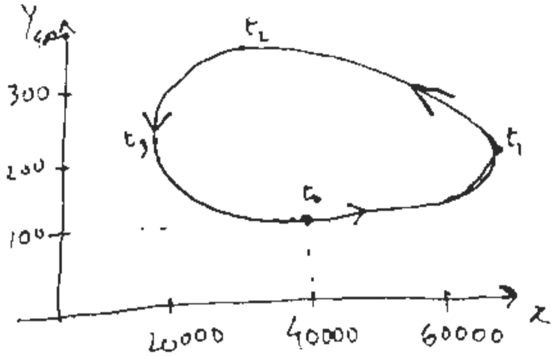
b) Equilibrium at $0.4x = .002xy \Leftrightarrow y = 200$

$0.2y = 0.000008xy \Leftrightarrow x = 25000$

When there are 200 birds and 25000 insects the populations are in equilibrium.

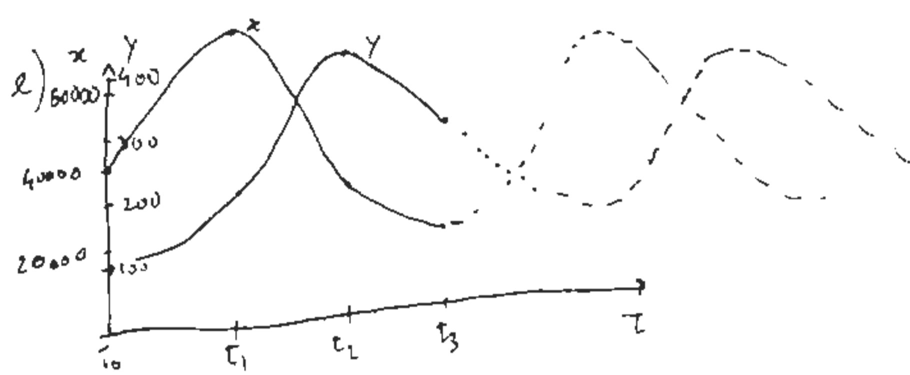
c) $\dfrac{dy}{dx} = \dfrac{-0.2y + 0.000008xy}{0.4x - 0.002xy}$

d)



First, increases the insect population, and slightly the birds. Then the birds start increasing faster and the insects start decreasing till a point when the bird population decreases and then the insect population increases and...

e)



21. a) Increases logistically.

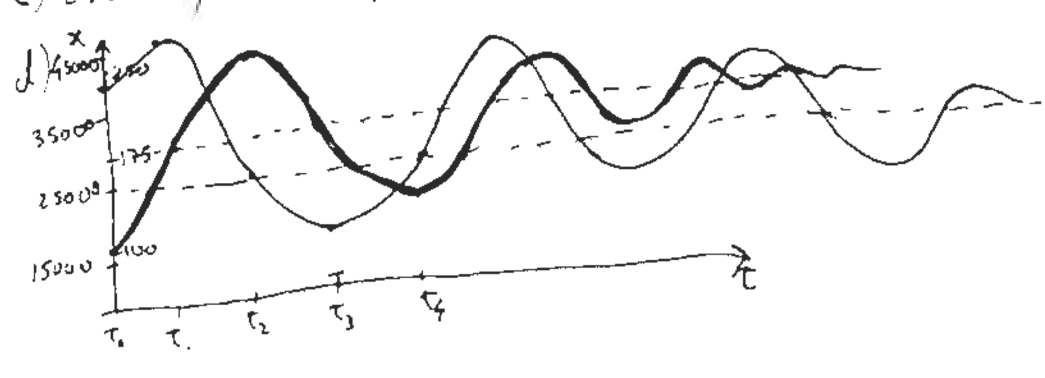
b) $x = 25000$

$y = .4(1 - 0.000005x) \Leftrightarrow y = 175$

equilibrium points: insects: 25000
birds = 175

c) Eventually they stay at their equilibrium points

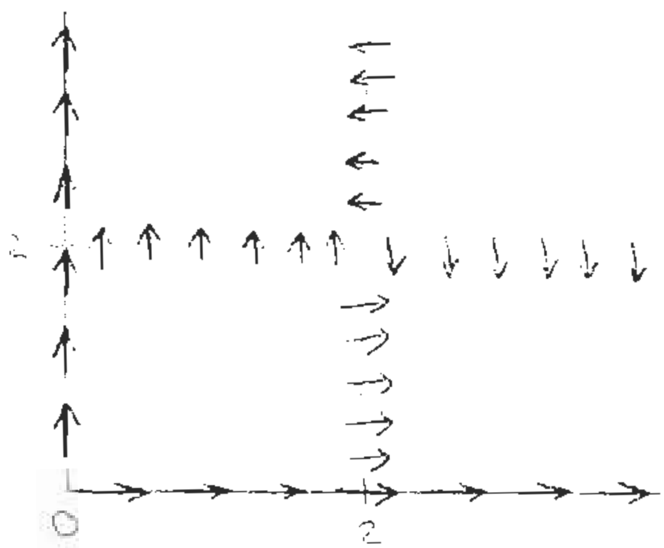
d)



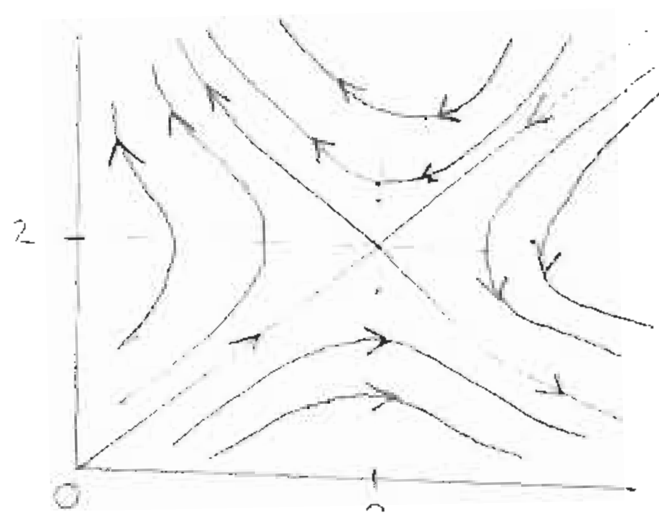
D Corrected phase portrait

d) Nullclines $x = 0$, $y = 0$ $x = 2$, $y = 2$

e) Tangents



f) Trajectories



15a) $\frac{dx}{dt} = x - x^2 - axy$ The two species are competitors. Each, in the absence of the other, grows logistically. There's a carrying capacity of 1 for either alone.

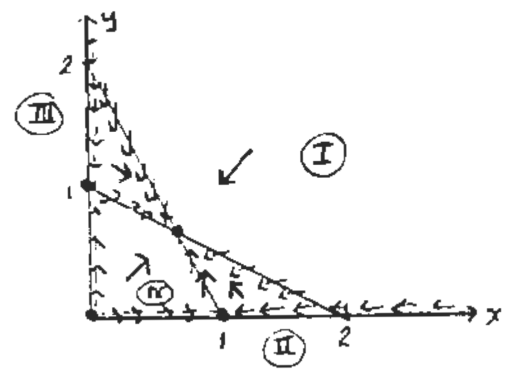
$\frac{dy}{dt} = y - y^2 - axy$.

b) $\boxed{\text{PHASE PLANE ANALYSIS WITH } a = \tfrac{1}{2}}$

$\frac{dx}{dt} = x(1 - x - \frac{1}{2}y)$ x-null clines: $x=0$ or $1-x-\frac{1}{2}y=0$ ie $y=2-2x$ vertical slope marks

$\frac{dy}{dt} = y(1-y-\frac{1}{2}x)$ y-null clines $y=0$ or $1-y-\frac{1}{2}x=0$, ie. $y=1-\frac{1}{2}x$ horizontal slope lines

The realistic modeling constraints restrict us to the 1st quadrant: $x \geq 0$, $y \geq 0$



Label regions in the 1st quadrant I, II, III, and IV as indicated

Equilibrium pts: where x-null clines & y-null clines intersect: (0,0) (1,0) (0,1) and $(\frac{2}{3}, \frac{2}{3})$

Interpreting the phase-plane analysis:

- suppose $y_0 = 0$ (we're on the x-axis). Then $y=0$ for all t. If $x_0 > 1$ then x decreases with t and approaches 1.
  If $x_0 < 1$ and $x_0 > 0$ then x increases with t and approaches 1.
- Similarly, if $x_0 = 0$ then $x=0$ for all t.
  If $y_0 > 1$ then y decreases with t & $y \to 1^+$
  If $0 < y_0 < 1$, then y increases with t & $y \to 1^-$
  (The x and y-axes - being trajectories, can't be crossed!)

- Suppose (x_0, y_0) is in region I.
  The trajectory heads down & left throughout the region.
  Therefore $(x(t), y(t))$ can tend towards $(\frac{2}{3}, \frac{2}{3})$ or will enter regions II or III. See analysis below for what happens next.
- Suppose (x_0, y_0) is in region II.
  The trajectory heads down and right throughout the region.
  The orientation of the slope marks on the null clines bounding region II indicate that the trajectory can't leave region II once it enters it. $(x(t), y(t))$ approaches $(\frac{2}{3}, \frac{2}{3})$.
- Suppose (x_0, y_0) is in region III.
  The trajectory heads up and left throughout the region and can't leave the region. $(x(t), y(t))$ approaches $(\frac{2}{3}, \frac{2}{3})$.
- Suppose (x_0, y_0) is in region IV.
  The trajectory heads up and right throughout the region.
  Therefore $(x(t), y(t))$ heads for $(\frac{2}{3}, \frac{2}{3})$ or enters region II or region III and from there heads to $(\frac{2}{3}, \frac{2}{3})$.

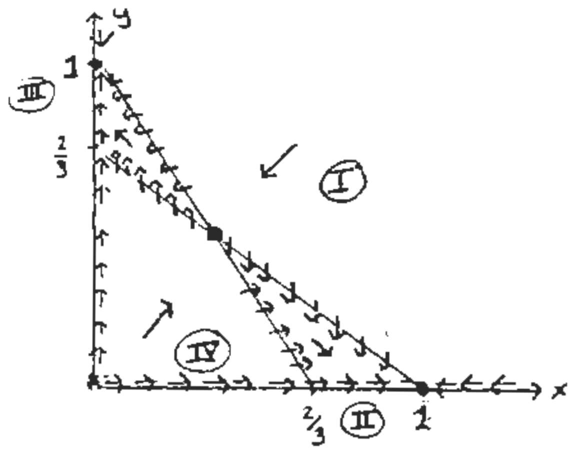
So, if $x_0 > 0$ and $y_0 > 0$, the system will tend towards the equilibrium $(\frac{2}{3}, \frac{2}{3})$.

If $\begin{cases} y_0 = 0 \\ x_0 > 0 \end{cases}$ $x(t) \to 1$. If $\begin{cases} x_0 = 0 \\ y_0 > 0 \end{cases}$ $y(t) \to 1$. If $\begin{matrix} x_0 = 0 \\ y_0 = 0 \end{matrix}$ then $\begin{matrix} x(t) = 0 \\ y(t) = 0 \end{matrix}$

$\boxed{\text{PHASE PLANE ANALYSIS WITH } a = 3/2}$

$\frac{dx}{dt} = x(1-x-\frac{3}{2}y)$ x-nullclines: $x=0$ or $1-x-\frac{3}{2}y=0$ i.e. $y=\frac{2}{3}-\frac{2}{3}x$ vertical slope marks

$\frac{dy}{dt} = y(1-y-\frac{3}{2}x)$ y-nullclines: $y=0$ or $1-y-\frac{3}{2}x=0$ i.e. $y= -\frac{3}{2}x$ horizontal slope marks



Equilibrium points: where x-nullclines and y nullclines intersect
$(1,0)$, $(0,1)$, $(0,0)$ and $(2/5, 2/5)$

Interpreting the phase-plane analysis

- Suppose $y_0 = 0$ (we're on the x-axis). Then $y(t)=0$ and $x(t)\to 1$ (decreasing if $x_0>1$ and increasing if $x_0<1$)
- Suppose $x_0=0$. Then $x(t)=0$ and $y(t)\to 1$.

- Suppose (x_0, y_0) is in region I.
  The trajectory heads down and left throughout the region
  $(x(t),y(t))$ either approaches $(2/5, 2/5)$ or will enter regions II or III. See analysis below.
- Suppose (x_0, y_0) is in region III.
  The trajectory heads up and left throughout the region and can't leave the region.
  $(x(t), y(t))$ approaches $(0,1)$.
- Suppose (x_0, y_0) is in region II.
  The trajectory heads down and right throughout the region & can't leave the region. $(x(t), (y(t))$ approaches $(1,0)$.
- Suppose (x_0, y_0) is in region IV.
  the trajectory heads up and right throughout the region.
  $(x(t), y(t))$ either approaches $(2/5, 2/5)$ or will enter region II and go towards $(1,0)$ or will enter region III and tend towards $(0,1)$.

Compare & contrast: For the case that there is either $x=0$ or $y=0$ the two systems behave the same.

For $x_0>0$ and $y_0>0$ the system with $a=\frac{1}{2}$ tends towards the equilibrium $(\frac{2}{3}, \frac{2}{3})$

For $x_0>0$ and $y_0>0$ the system with $a=3/2$ tends towards $(1,0)$ or $(0,1)$ or $(2/5, 2/5)$

it is not necessary to say this to do the problem - but it's true. (You can argue this using symmetry & uniqueness.)

In fact, if $x_0 \neq y_0$ then the trajectory tends to $x=1$ $y=0$ if $x_0 > y_0$ and to $x=0$, $y=1$ if $x_0 < y_0$. In other words - one "species" will win out and the other will become extinct - in accordance with Darwin's principle of Competitive exclusion.