Name: ______________________ ID#: ______________

Solutions to Midterm I

Math 1b
Calculus, Series, Differential Equations

22 October 2003

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

Please circle your section:

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Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.
—Handbook for Students

Problem Number	Possible Points	Points Earned
1	12	
2	10	
3	20	
4	10	
5	15	
6	15	
7	18	
Total	100	

1. (12 Points) *Compute the following limits, if they exist.*

Note. Each part was given four points; two points for the correct limit, two for the correct work shown.

(a) $\lim_{n\to\infty} \frac{\ln(2+e^n)}{3n}$

Solution. We can use L'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{\ln(2+e^n)}{3n} \overset{\text{H}}{=} \lim_{n\to\infty} \frac{\frac{e^n}{2+e^n}}{3} = \lim_{n\to\infty} \frac{3e^n}{2+e^n} = \lim_{n\to\infty} \frac{3}{2e^{-n}+1} = 3.$$

□

(b) $\lim_{n\to\infty} \left(\frac{n+2}{n}\right)^{3n}$

Solution. Let us manipulate the limit.

$$\lim_{n\to\infty} \left(\frac{n+2}{n}\right)^{3n} = \lim_{n\to\infty} e^{3n\ln\left(\frac{n+2}{n}\right)} = e^{\lim_{n\to\infty} 3n\ln\left(\frac{n+2}{n}\right)},$$

provided the last limit exists. We can find it with L'Hôptial's Rule again:

$$\begin{aligned} \lim_{n\to\infty} 3n\ln\left(\frac{n+2}{n}\right) &= \lim_{n\to\infty} \frac{\ln(n+2)-\ln n}{\frac{1}{3n}} \\ &\overset{\text{H}}{=} \lim_{n\to\infty} \frac{\frac{1}{n+2}-\frac{1}{n}}{-\frac{1}{3n^2}} = \lim_{n\to\infty} \frac{6n^2}{n(n+2)} \\ &= 6. \end{aligned}$$

So the limit we want is e^6. □

(c) $\lim_{n\to\infty} \frac{\cos(n)}{\ln(n)}$.

Solution. Notice that

$$0 \le \left|\frac{\cos n}{\ln n}\right| \le 1\frac{1}{\ln n}.$$

The right-hand side of this inequality goes to zero as $n \to \infty$, so $\lim_{n\to\infty} \left|\frac{\cos(n)}{\ln(n)}\right| = 0$. It is then a consequence of the Squeeze Theorem that $\lim_{n\to\infty} \frac{\cos(n)}{\ln(n)} = 0$ as well. □

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2. (10 Points) *In January of 2003, Harvard Dining Services bought 100 eight-ounce cans of soup. In February, the factory decreased the amount of soup per can by 1% to 7.92 ounces, and HDS compensated by ordering 1% more cans (i.e., 101 cans). Assume that this pattern continues from month to month, that is, each month the can size decreases by 1% and the number of cans ordered increases by 1%*[1]*, and that the soup is never eaten. How many ounces of soup are on hand after infinitely many months?*

Solution. The number of cans of soup ordered in month n is $100(1.01)^{n-1}$. The number of ounces per can in month n is $8(0.99)^{n-1}$. Therefore, the number of ounces of soup in month n is

$$c_n = (100)(1.01)^{n-1}(8)(0.99)^{n-1} = 800(0.9999)^{n-1}.$$

Getting this expression usually earned half credit. Adding them up, we have a total number of ounces of soup

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} 800(0.9999)^{n-1} = \frac{800}{1 - 0.9999} = 8,000,000$$

on hand. □

[1]Yes, this means that HDS starts ordering a non-integer number of cans eventually. We'll allow this in our theoretical universe.

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3. (20 Points) *Determine whether the following series are convergent or divergent. In the case of a convergent series with negative terms in it, determine whether the convergence is absolute. Indicate clearly which tests you use and what conclusions you draw from them.*

Note. Each convergence problem was worth four points ((b) is actually two problems). For each, one point was given for the correct answer, one for the name of the test used, and two for showing the correct work.

(a) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

Solution. The ratio of successive terms is

$$\frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}} = \frac{n+1}{n}\frac{1}{e} \to \frac{1}{e}$$

as $n \to \infty$, so by the Ratio Test the series converges. □

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{(n+1)(n+2)}$

Solution. Let $b_n = \frac{n}{(n+1)(n+2)} = \frac{n}{n^2+3n+2}$. We need to check that $b_n > 0$ for all n (yes), that $\lim_{n\to\infty} b_n = 0$ (also clear since the numerator is a polynomial of degree 1 and the denominator is a polynomial of degree 2), and that $\{b_n\}$ is decreasing. To see the last part, notice

$$\frac{d}{dx}\frac{x}{x^2+3x+2} = \frac{(x^2+3x+2)(1) - x(2x+3)}{(x^2+3x+2)^2} = \frac{-x^2+2}{(x^2+3x+2)^2},$$

which is negative as long as $\sqrt{x} > 2$. So the sequence is (eventually) decreasing. Therefore, the Alternating Series Test says that the given series $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ converges.

The absolute-value series $\sum_{n=1}^{\infty} b_n$ is divergent. To see this, notice

$$\frac{b_n}{\frac{1}{n}} = \frac{n^2}{(n+1)(n+2)} \to 1$$

as $n \to \infty$. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so then by the Limit Comparison Test must $\sum_{n=1}^{\infty} b_n$. Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{(n+1)(n+2)}$ is *not* absolutely convergent. □

(c) $\sum_{n=1}^{\infty} 2^{1/n}$.

Solution. Notice that

$$2^{1/n} = e^{\frac{1}{n}\ln 2} \to e^0 = 1,$$

so the Test For Divergence indicates the series diverges. □

(d) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$.

Solution. Notice that

$$\frac{\frac{e^{1/n}}{n^2}}{\frac{1}{n^2}} = e^{1/n} \to 1,$$

so by the Limit Comparison Test the series converges. □

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4. (10 Points) *Let* $A(x)$ *be the* Airy function

$$A(x) = 1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots .$$

(a) If $A(x) = \sum_{n=0}^{\infty} c_n x^{3n}$*, what is* c_n*?*

Solution. Notice that not all of the integers up to $3n$ are represented in the denominator of c_n! So although $\frac{1}{(3n)!}$ is close, it is not right. The most succinct way to write it is

$$c_n = \frac{1}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)}$$

This was a four-point part. □

(b) Find the interval of convergence of $A(x)$*.*

Solution. We can use the Ratio Test:

$$\left|\frac{c_{n+1}x^{3n+3}}{c_n x^{3n}}\right| = |x|^3 \frac{\frac{1}{2\cdot3\cdot5\cdot6\cdots(3n-1)(3n)(3n+2)(3n+3)}}{\frac{1}{2\cdot3\cdot5\cdot6\cdots(3n-1)(3n)}} = \frac{|x^3|}{(3n+2)(3n+3)} \to 0,$$

as $n \to \infty$. Thus the series converges for all x, and so the interval of convergence is $(-\infty, +\infty)$.

This was a six-point part: Two points were awarded for getting the right ratio, two for computing the right limit, and two for converting that to the right answer. Points were deducted for not giving the interval of convergence. □

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5. (15 Points) *Show that* $(x^2 + x)e^x = \sum_{n=1}^{\infty} \frac{n^2 x^n}{n!}$ *for all* x.

Solution. Remember that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Then

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$
$$x^2e^x = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!}$$

So the sum is

$$(x + x^2)e^x = x + \sum_{n=2}^{\infty} \left(\frac{1}{(n-1)!} + \frac{1}{(n-2)!} \right) x^n.$$

Now for $n \geq 2$,

$$\frac{1}{(n-1)!} + \frac{1}{(n-2)!} = \frac{1}{(n-1)!} + \frac{n-1}{(n-1)!} = \frac{n}{(n-1)!} = \frac{n^2}{n!},$$

and by inspection we see that the coefficient on x^1 is $\frac{1^2}{1!}$ as well. So

$$(x + x^2)e^x = \sum_{n=1}^{\infty} \frac{n^2 x^n}{n!}$$

as desired. □

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6. (15 Points) *We will approximate* $\frac{1}{\sqrt[3]{1.1}}$ *with an error no bigger than* 10^{-4}.

(a) Write the number as a series.

Solution. The Binomial Theorem is really the way to go here.

$$\frac{1}{\sqrt[3]{1.1}} = \left(1 + \tfrac{1}{10}\right)^{-1/3} = \sum_{n=0}^{\infty} \binom{-1/3}{n} \left(\frac{1}{10}\right).$$

Points were deducted for getting the wrong exponent; it transformed the problem into another one albeit pretty equally hard. Points were also deducted for leaving the answer as a power series in x; this is not what was asked for. The part was worth four points altogether.

You could get at the answer by computing the Taylor Series for $f(x) = x^{-1/3}$ at 1 (Computing it at zero won't work because the function is not even differentiable there.) You basically have to find an expression for $f^{(n)}(1)$ in terms of n, and that's precisely $\binom{-1/3}{n}$.

Writing "..." does not make a proof. □

(b) Show that the terms in the series are alternating in sign.

Solution. This part was worth three points. Just writing out a few terms and pointing to the alternating signs does not constitute a proof; how do we know they don't become all positive after a million terms? This answer typically got one point.

We can compute $\binom{-1/3}{n}$ explicitly:

$$\begin{aligned}\binom{-1/3}{n} &= \frac{1}{n!}\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-1\right)\cdots\left(-\frac{1}{3}-n+1\right)\\ &= \frac{1}{n!}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\cdots\left(-\frac{3n-2}{3}\right)\\ &= \frac{(-1)^n 1\cdot 4\cdot 7\cdots(3n-2)}{n!3^n}.\end{aligned}$$

So

$$\frac{1}{\sqrt[3]{1.1}} = \sum_{n=0}^{\infty} \frac{(-1)^n 1\cdot 4\cdot 7\cdots(3n-2)}{n!3^n 10^n}. = \sum_{n=-}^{\infty} a_n.$$

Every factor in a_n other than the $(-1)^n$ factor is positive, so the terms alternate in sign.

Another way to see this is to show that

$$a_{n+1} = \frac{-1/3-n}{10(n+1)} a_n,$$

and this multiplier is always negative. So the signs alternate. □

(c) Assume that the other conditions for the Alternating Series Estimation Theorem are satisfied as well. Use it to estimate the sum of the series with the desired accuracy. (You can leave your answer as a fraction or a sum of fractions.)

Solution. The Alternating Series Estimation Theorem says that

$$|s - s_n| < |a_{n+1}|.$$

So if we choose n such that $a_{n+1} < 10^{-4}$, we know s_n is close enough to s.

n	a_n
0	1
1	$\left(-\frac{1}{3}\right)\left(\frac{1}{0}\right) = \dfrac{1}{30}$
2	$\frac{1}{2}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(\frac{1}{100}\right) = \dfrac{2}{900} = \dfrac{1}{450}$
3	$\frac{1}{3!}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\left(\frac{1}{1000}\right) = \dfrac{-28}{162,000}$
4	$\frac{1}{4!}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\left(-\frac{11}{3}\right)\left(\frac{1}{10,000}\right) = \dfrac{308}{24 \cdot 81 \cdot 10^4}.$

Now $24 \times 81 = 1944$, and $\frac{308}{1944} < 1$, so $a_4 < 10^{-4}$. This means that we need only sum up to the $n = 3$ term.

$$\frac{1}{\sqrt[3]{1.1}} \approx 1 - \frac{1}{30} + \frac{1}{450} - \frac{28}{162,000}.$$

□

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7. (18 Points) *Label each of the following statments as true (**T**) or false (**F**). If the statement is true, explain why. If the statement is false, explain why or give an example that disproves the statement.*

Do not assume more than is stated in the question. For instance, do not assume that sequences consist of all positive terms or that limits exist.

Note. Each of these was three points: one for the correct letter (T/F), and two for a valid justification.

______ *(a) The series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ *converges.*

Solution. This statement is true. It is a simple application of the Alternating Series Test. We need only check that $\left\{ \frac{1}{\sqrt{n}} \right\}$ is a positive sequence which decreases to zero, and it is. ☐

______ *(b) The series* $\sum_{n=1}^{\infty} \sin(n)x^n$ *converges if* $|x| < 1$.

Solution. This is true. Unfortunately, none of our powerful tests say so. This is because neither of the limits

$$\lim_{n\to\infty} \left| \frac{\sin(n+1)}{\sin n} x \right|$$

nor

$$\lim_{n\to\infty} |\sin(n)|^{1/n} |x|$$

exist. One can only argue that since

$$0 \le |\sin(n)x^n| \le |x|^n ,$$

we must have our given series less than a geometric series, which converges when $|x| < 1$. So the series converges if $|x| < 1$. [We also know that if $x = \pm 1$, the terms do not approach zero, so the power series diverges at ± 1. Hence the radius of convergence is actually 1.] ☐

______ *(c) If* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\sum_{n=1}^{\infty} a_n^2$ *is convergent.*

Solution. This is false. Consider the series given in part (a). We have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}},$$

which converges, yet

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. □

______ *(d) If $\{a_n\}$ is a positive sequence and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} \ln(a_n)$ converges.*

Solution. Most students who missed this forgot what $\lim_{x\to 0} \ln(x)$ is. Think about it: as x goes to zero, it passes through $\frac{1}{e}, \frac{1}{e^2}, \frac{1}{e^3}, \ldots$. These have natural logarithms of $-1, -2, -3, \ldots$. Hence

$$\lim_{x\to 0} \ln(x) = -\infty.$$

Now since $\sum_{n=1}^{\infty} a_n$ converges, we must have $\lim_{n\to\infty} a_n = 0$. Therefore, $\lim_{n\to\infty} \ln(a_n) = -\infty$. So by the Test for Divergence, $\sum_{n=1}^{\infty} \ln(a_n)$ diverges. The statement is False. □

______ *(e) If f is a continuous function and $a_n = f(n)$ and $\int_0^{\infty} f(x)\, dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Solution. This is False. A good counterexample is the function $f(x) = \frac{\cos(2\pi x)}{x}$, whose graph bounces between the graphs of $\frac{1}{x}$ and $-\frac{1}{x}$. It can be shown that the integral

$$\int_1^{\infty} \frac{\cos(2\pi x)\, dx}{x}$$

converges, but $f(n) = \frac{\cos(2\pi n)}{n} = \frac{1}{n}$ for all n, so the sum $\sum f(n)$ cannot converge. □

______ *(f) If $\sum_{n=1}^{\infty} c_n x^n$ converges at $x = 1$, then $\sum_{n=1}^{\infty} n c_n x^{n-1}$ converges at $x = 1$.*

Solution. This too is false. The second power series is the derivative of the first power series, so they both have the same radius of convergence R. The fact that the first converges at 1 gives that $R < 1$, but that alone does not guarantee that the second power series converges at 1. It could be the case that $R = 1$.

A counterexample has just that problem: Try $c_n = \frac{1}{n^2}$. Then $\sum_{n=1}^{\infty} c_n 1^n$ converges, but

$$\sum_{n=1}^{\infty} n c_n 1^{n-1} = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. □

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