Solutions to the Second Exam for Mathematics 1b

April 10, 2003

1. (a) Since all the terms of the series $\sum_{n+1}^{\infty} \frac{5}{2^n+n}$ are positive, we can use comparison to show convergence. Note that the geometric series $\sum_{n=1}^{\infty} \frac{5}{2^n}$ converges, since the ratio is $\frac{1}{2}$ and $\left|\frac{1}{2}\right| < 1$. Moreover, when we compare terms we see that

$$\frac{5}{2^n+n} < \frac{5}{2^n} \quad \text{for all n}$$

Therefore, by the (direct) comparison test, the original series converges.

The ratio test would also demonstrate convergence, because $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1$. The limit comparison could be used as well, comparing to either

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{5}{2^n}$$

(b) The ratio of the terms is

$$\left|\frac{a_n+1}{a_n}\right| = \left|\frac{(n+1)^4+1}{4^{n+1}} \cdot \frac{4^n}{n^4+1}\right| = \left(\frac{(n+1)^4+1}{n^4+1}\right)\left(\frac{4^n}{4^{n+1}}\right) = \frac{1}{4}\left(\frac{(n+1)^4+1}{n^4+1}\right)$$

so the limit is

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{4}\left(\frac{(n+1)^4+1}{n^4+1}\right) = \frac{1}{4}\lim_{n\to\infty}\left(\frac{(1+\frac{1}{n})^4+\frac{1}{n^4}}{1+\frac{1}{n^4}}\right) = \frac{1}{4} < 1$$

Therefore we can conclude by the ratio test that the series $\sum_{n=1}^{\infty}\frac{n^4+1}{4^n}$ converges.

In order to consider the limit of the *terms*, one would need to use L'Hopital's rule. The limit $\lim_{n\to\infty}\frac{n^4+1}{4^n} = 0$ even though the first few terms are increasing.

(c) Even though the series $\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{2n}$ is an alternating series with the terms decreasing ($|a_{n+1}| < |a_n|$), it diverges by the Nth term divergence test—the limit of the terms is

$$\lim_{n\to\infty}\left|(-1)^n\frac{n+1}{2n}\right| = \frac{1}{2} \neq 0$$

(d) The terms of the series $\sum_{n=1}^{\infty}\frac{\sin k}{k^3+\sqrt{k}}$ are not all positive but neither do they alternate. Therefore, consider the absolute value of the terms.

$$\left|\frac{\sin k}{k^3+\sqrt{k}}\right| \leq \frac{1}{k^3}$$

since $|\sin k| \leq 1$. But we know that the series $\sum_{n=1}^{\infty}\frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$), so $\sum_{n=1}^{\infty}\frac{\sin k}{k^3+1}$ converges as well.

Even though $\sin k$ oscillates and $\lim_{n\to\infty}\sin k$ does not exist, this series can still converge because $\sin k$ is bounded by 1 and the denominator is growing with respect to k. We can check that that the limit of the terms, $\lim_{n\to\infty}\frac{|\sin k|}{k^3+\sqrt{k}}$ is indeed zero by sqeezing $\frac{|\sin k|}{k^3+\sqrt{k}}$ between 0 and $\frac{1}{k^3}$.

(e) Again, the Nth term divergence test will give us divergence here. Note that $\ln n < n$ for all n (think of the graphs $y = x$ and $y = \ln x$, or consider what $\ln n$ means). Therefore, $\frac{n}{\ln n} > 1$ for all n, and so $\lim_{n\to\infty}\frac{n}{\ln n} \neq 0$. In fact, we can use L'Hospital's rule to give us

$$\lim_{n\to\infty}\frac{n}{\ln n} = \lim_{n\to\infty}\frac{1}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} n \to \infty$$

and the series must diverge.

2. (a) The series will either converge only at $x = -3$, for all x, or for all x in some interval centered (give or take the endpoints) around $x = -3$.

Use the Ratio Test for starters: $\lim_{k\to\infty} \frac{\frac{|x+3|^{k+1}}{(k+1)4^{k+1}}}{\frac{|x+3|^k}{(k)4^k}} = \lim_{k\to\infty} \frac{|x+3|^{k+1}}{(k+1)4^{k+1}} \cdot \frac{(k)4^k}{|x+3|^k} = \lim_{k\to\infty} \frac{|x+3|k}{(k+1)4} = \frac{|x+3|}{4}$

The series converges whenever $\frac{|x+3|}{4} < 1$, i.e. whenever $|x+3| < 4$, and diverges whenever $|x+3| > 4$. In other words, the series converges for all x at a distance less than 4 from -3. Draw a picture; the series converges on $-7 < x < 1$.

Now test the endpoints:

At $x = -7$ we get $\sum_{k=3}^{\infty} \frac{(-4)^k}{k4^k} = \sum_{k=3}^{\infty} \frac{(-1)^k}{k}$. This is the alternating harmonic series. It converges by the alternating series test: the series is alternating, the terms are decreasing in magnitude, and the terms are tending towards zero.

At $x = 1$ we get $\sum_{k=3}^{\infty} \frac{(4)^k}{k4^k} = \sum_{k=3}^{\infty} \frac{1}{k}$. This is the harmonic series (a p-series with $p = 1$. It diverges.

The interval of convergence is $[-7, 1)$.

(b) The series given is centered at $x = 2$. Therefore the interval of convergence will be symmetric about 2, except possibly at the endpoints. It converges for $x = 4$, therefore its radius of convergence is at least 2. It diverges for $x = 5.5$, so the radius of convergence is at most 3.5.

The interval of convergence is at least $(0, 4]$ and at most $[-1.5, 5.5)$.

Therefore, the series converges when $x = 1$, diverges when $x = -4$ and there is not enough information to determine whether or not it converges at $x = 0$. It's possible that $x = 0$ could be an endpoint of integration and not be included in the interval of integration. On the other hand, it could be an included endpoint, or the radius could be larger than 2.

Note: we don't know any specifics about a_n, so we can't say that the series at $x = 4$, $\sum_{n=1}^{\infty} a_n 2^n$ converges absolutely. It is possible that this series is actually alternating, for instance.

3. (a) The Taylor series about $u = 0$ generated by $\sin u$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}.$$

Substitute $u = x^3$ to get the Taylor series for $\sin(x^3)$, and then multiply by x to get the Taylor series for $x\sin(x^3)$: we obtain

$$\sum_{n=0}^{\infty} x(-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!},$$

which we can rewrite as

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{(2n+1)!}.$$

(b) The Taylor series for $\sin u$ converges to $\sin u$ for all u. The series above therefore converges to $x\sin(x^3)$ for all x. The radius of convergence is infinite.

(c) We can integrate the series term by term, so

$$\int_0^{0.1} x\sin(x^3)\,dx = \left[\sum_0^{\infty} (-1)^n \frac{x^{6n+5}}{(6n+5)(2n+1)!}\right]_0^{0.1}.$$

Thus the integral is the sum of the series

$$\frac{10^{-5}}{5 \times 1!} - \frac{10^{-11}}{11 \times 3!} + \frac{10^{-17}}{17 \times 5!} - \cdots .$$

The series is alternating and satisfies the conditions of the alternating series test (terms tending to zero and decreasing in magnitude). From this it follows that, if we truncate the series and evaluate the nth partial sum, then the absolute value of the error is no larger than the absolute value of the first neglected term. The third term above is already smaller than 10^{-14}. So

$$\int_0^{0.1} x\sin(x^3)\,dx \approx \frac{10^{-5}}{5} - \frac{10^{-11}}{66}$$

with an error of (much) less than 10^{-14}.

2. First, note the information we are given. The series $\sum_{n=1}^{\infty} a_n = 0.9$, which means the series converges to 0.9. From this we know that $lim_{n\to\infty} a_n = 0$ or else the series would diverge. We are also told that the terms a_n are definitely positive for $n > 5$. We know nothing at all about the terms a_1, a_2, a_3, a_4 and a_5 but the first few terms of a series do not affect its convergence—only its sum. Now to the claims:

   (a) $\lim\limits_{n\to\infty} a_n = 0.9$

   This must be FALSE, because as noted above, we must have $\lim_{n\to\infty} a_n = 0$ for convergence of the series.

   (b) $\lim\limits_{n\to\infty} s_n = 0.9$

   TRUE—this is what it means for a series to converge, since the $s_n = a_1 + a_2 + \ldots + a_n$ are the partial sums and

   $$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k\right) = \lim_{n\to\infty} s_n$$

   .

   (c) $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ converges.

   This series is not necessarily strictly an alternating series because we do not know anything about the first five terms. So let us consider instead the series $\sum\limits_{n=6}^{\infty}(-1)^n a_n$. These two series will either both converge or both diverge because the first few terms of a series do not affect its convergence. Now, our new series is guaranteed to be alternating. However, since the terms are not guaranteed to be decreasing in magnitude (we don't know if $|a_{n+1}| < |a_n|$ or not) we cannot use the alternating series test—it fails to give us any information. Instead, consider the absolute convergence of this new series:

   $$\sum_{n=6}^{\infty} |(-1)^n a_n| = \sum_{n=6}^{\infty} a_n \quad \text{converges}$$

   so $\sum\limits_{n=6}^{\infty}(-1)^n a_n$ converges absolutely, and therefore converges. Since adding five more terms does not affect convergence, $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ converges as well, and the statement is TRUE.

   (d) If $b_n = a_n + 0.1$ then $\sum\limits_{n=1}^{\infty} b_n$ converges.

   FALSE—if $b_n = a_n + 0.1$ then

   $$\lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n + 0.1) = \left(\lim_{n\to\infty} a_n\right) + 0.1 = 0 + 0.1 = 0.1 \neq 0$$

   so by the Nth term divergence test, $\sum\limits_{n=1}^{\infty} b_n$ must diverge.

(e) $\sum_{n=1}^{\infty} na_n$ converges.

With this statement, it is IMPOSSIBLE TO TELL if it is true or false. It depends on the value of the a_n. For example, if $a_n = \frac{C}{n^2}$ where C is some constant to make the sum equal 0.9, then the series $\sum_{n=1}^{\infty} na_n$ will not converge. However, if $a_n = \frac{C}{n^3}$ then the series will converge.
Note: n is not a constant in this situation—the sigma notation means

$$\sum_{n=1}^{\infty} na_n = 1a_1 + 2a_2 + 3a_3 + 4a_4 + \dots$$

so it makes no sense to try to pull n out of the sum.

(f) $\sum_{n=1}^{\infty} (a_n)^{2003}$ converges.

This is TRUE. For $n > 5$ we know a_n is positive, so $(a_n)^{2003}$ is positive as well. Since $\lim_{n\to\infty} a_n = 0$, a_n must eventually be less than one (We could say there is some K, such that $0 < a_n < 1$ if $n \geq K$). If $0 < a_n < 1$ then $0 < (a_n)^{2003} < a_n$ and we can use comparison. Again, the first few terms of the sequence do not affect convergence. Since $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=K}^{\infty} a_n$ converges and then by comparison $\sum_{n=K}^{\infty} (a_n)^{2003}$ converges so $\sum_{n=1}^{\infty} (a_n)^{2003}$ must converge.

(g) $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}$ converges.

This is TRUE as well. For $n > 5$ both a_n and $\frac{a_n}{\sqrt{n}}$ are positive, so we can use comparison. As long as they are positive, $\frac{a_n}{\sqrt{n}} < a_n$ since $\sqrt{n} > 1$. So as above, the fact that $\sum_{n=1}^{\infty} a_n$ converges implies that $\sum_{n=6}^{\infty} a_n$ converges, and therefore $\sum_{n=6}^{\infty} \frac{a_n}{\sqrt{n}}$ converges by comparison, and $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}$ must converge.

5. (a) We apply the ratio test: $\sum_{n=1}^{\infty} n2^n x^{n-1}$ converges if the limit of the ratio of consecutive terms is less than 1:

$$\lim_{n\to\infty} \left| \frac{(n+1)2^{n+1}x^n}{n2^n x^{n-1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} 2x \right| = 2|x| < 1$$

$$|x| < \frac{1}{2}$$

so the radius of convergence is 1/2. (Note that we don't yet know what happens for $x = \pm 1/2$, but the question does not ask for this.)

(b) The easy way to find the Maclaurin series for $\frac{1}{1-2x}$ is to take the geometric series:

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n \tag{1}$$

and substitute $u = 2x$:

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n. \tag{2}$$

(Some people did the problem by repeated differentiating; this also works, but takes longer.) Since we know that the geometric series (**??**) converges only for $|u| < 1$, the new series (**??**) coverges only for $|2x| < 1$ or $|x| < 1/2$, so the interval of convergence is $(-1/2, 1/2)$ (with neither endpoint included). You can also use the ratio test and check the two endpoints to get the same conclusion.

(c) Differentiate the series in equation **??**:

$$\frac{d}{dx}\left(\frac{1}{1-2x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} 2^n x^n\right)$$
$$= \sum_{n=0}^{\infty} n2^n x^{n-1}$$

which is the equation from part (a). Therefore

$$\sum_{n=0}^{\infty} n2^n x^{n-1} = \frac{d}{dx}\left(\frac{1}{1-2x}\right) = \frac{2}{(1-2x)^2}.$$

6. (a) One way to find the Taylor series of $1/x^2$ about $x = -1$ is by simply computing its derivatives, evaluate them at -1, and use the formula for the Taylor series. Another method is to start with the geometric series:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

and to replace x by $1 + x$ (in order to get a series expansion in powers of 1+x):

$$-\frac{1}{x} = \sum_{k=0}^{\infty} (x+1)^k.$$

Notice that the derivative of $-1/x$ is $1/x^2$, so taking the derivative of the series on the right hand side term by term we conclude:

$$\frac{1}{x^2} = \sum_{k=1}^{\infty} k(x+1)^{k-1},$$

which is the Taylor series about $x = -1$.

(b) Apply the ratio test:

$$\lim_{k\to\infty} \left|\frac{(k+1)(x+1)^k}{k(x+1)^{k-1}}\right| = |x+1|,$$

so the series converges for $|x + 1| < 1$, that is $-2 < x < 0$. At both endpoints the series diverges by the n-th term test for divergence (the limit of the general term is not zero).

7. The second degree Taylor polynomial at $x = 2$ is given by $f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2$. At $x = 2$ we can see that the function is positive, increasing, and concave down.

Therefore, the constant term must be positive (f is positive at 2). So (a) can't be the second degree Taylor polynomial.

Similarly, the coefficient of the $(x - 2)$ term must be positive (f is increasing). This rules out (b).

The coefficient of the $(x - 2)^2$ term has to be negative, since f is concave down at $x = 2$. This rules out both (b) and (d). This leaves (b) as the correct answer.