Solutions to the Second Examination: Mathematics1b
April 13, 2005

1. Use the ratio test to determine the radius of convergence:

$$1 > \lim_{n\to\infty}\left|\frac{\frac{(x-3)^{n+1}}{5^{n+1}\sqrt{n+1}}}{\frac{(x-3)^n}{5^n\sqrt{n}}}\right| = \lim_{n\to\infty}\left|\frac{(x-3)}{5}\sqrt{\frac{n}{n+1}}\right| = \left|\frac{(x-3)}{5}\right|$$

$$\Rightarrow |x-3| < 5$$

Hence the interval of convergence is at least $(-2, 8)$, but we need to check the endpoints.

$$x = -2 \ : \ \sum_{n=0}^{\infty}\frac{(-5)^n}{5^n\sqrt{n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

This is an alternating series so let's check the conditions of the alternating series test:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$$

$$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

So the series converges for $x = 2$.

$$x = 8 \ : \ \sum_{n=0}^{\infty}\frac{5^n}{5^n\sqrt{n}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$$

This is a p-series with $p = 1/2$, and so diverges. Hence the series will not converge for $x = 8$.

Thus the interval of convergence is $[-2, 8)$.

2. (a) $\lim_{n\to\infty} s_n = 0$ must be false, since $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n = 0.6$ by assumption.
   (b) $a_{n+1} < a_n$ for all n can not be decided, since only a_n for large n are important for convergence.
   (c) $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1$ can not be decided, since for example $\sum\frac{1}{n^2}$ is convergent and $\lim_{n\to\infty}\frac{(n+1)^2}{(n^2)} = 1$.
   (d) $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} > 1$ must be false since ratio test would imply divergence.
   (e) $\sum_{n=1}^{\infty} s_n$ does not converge, since $\lim_{n\to\infty} s_n = 0.6$, hence failing the n-th term test.
   (f) $\sum_{n=1}^{\infty} a_n^2$ converges. This is true, since we have $a_n < 1$ by assumption, hence, $0 < a_n^2 < a_n$ for all n, and we find convergence by direct comparison.
   (g) $\sum_{n=1}^{\infty}\ln a_n$ converges. This must be false, since it follows from $\lim_{n\to\infty} a_n = 0$, that $\lim_{n\to\infty}\ln a_n = -\infty$, hence, it fails the n-th term test.

3. (16 points) Determine whether or not the series converges. You must tell us explicitly what convergence/divergence test you are applying and what criteria you used to draw your conclusion.

   Please note that no points were awarded for drawing the correct conclusion from incorrect applications of tests or incorrect limit computations. At least one point was awarded, however, for choosing a test that was appropriate for the problem.

   (a) $\sum_{n=3}^{\infty}\frac{5\sqrt{n}}{n^2+2n}$.
   One could apply either the comparison test or the limit comparison test with $5\sum_{n=3}^{\infty}\frac{1}{n^{\frac{3}{2}}}$. For the comparison test, one only needed to notice that

$$\frac{5\sqrt{n}}{n^2+2n} < \frac{5}{n^{\frac{3}{2}}}.$$

$\sum_{n=3}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a convergent p-series with $p = \frac{3}{2}$, so $\sum_{n=3}^{\infty} \frac{5\sqrt{n}}{n^2+2n}$ must converge.

A very common mistake was to compare the series to $\frac{1}{n^2}$ while ignoring the contribution of the numerator, namely $5\sqrt{n}$, which does go to infinity as $n \to \infty$. Of course, the fraction itself behaves like $\frac{5}{n^{\frac{3}{2}}}$ as $n \to \infty$.

(b) $\sum_{n=1}^{\infty} \frac{3n}{e^{n^2}}$

One could apply the comparison test with $\sum \frac{1}{e^n}$ (which is a convergent geometric series), the integral test, or the ratio test here. For the ratio test, one looks at

$$a_{n+1} = \frac{3(n+1)}{e^{(n+1)^2}} = \frac{3(n+1)}{e^{n^2+2n+1}}$$

and

$$a_n = \frac{3n}{e^{n^2}}.$$

Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3(n+1)}{e^{n^2+2n+1}} \frac{e^{n^2}}{3n} = \lim_{n\to\infty} \frac{n+1}{n} \frac{1}{e^{2n+1}} = 0 < 1$$

Thus, $\sum_{n=1}^{\infty} \frac{3n}{e^{n^2}}$ converges.

The most common mistake was failing to expand $(n+1)^2$ properly. For the integral test, some people had trouble taking the limit in order to compute the resulting improper integral. Another common mistake was to conclude that $\lim_{n\to\infty} \frac{3n}{e^{n^2}}$ was not equal to zero.

(c) $\sum_{n=3}^{\infty} \frac{\cos n}{3^n - 8}$

Here one has to apply the comparison test. The first observation is that $|\cos n| \leq 1$ for all n. Therefore,

$$\left| \frac{\cos n}{3^n - 8} \right| \leq \frac{1}{3^n - 8} < 2 \cdot \frac{1}{3^n}.$$

Since $\sum_{n=3}^{\infty} \frac{1}{3^n}$ is a convergent geometric series, $\sum_{n=3}^{\infty} \frac{\cos n}{3^n-8}$ converges absolutely, so it must converge. Some people used the limit comparison test to show $\sum_{n=3}^{\infty} \frac{1}{3^n-8}$ behaved the same way as $\sum_{n=3}^{\infty} \frac{1}{3^n}$ instead of setting up a comparison as above.

The common mistakes were bounding $\cos n$ from above but forgetting to bound it from below, assuming $\sum_{n=3}^{\infty} \frac{\cos n}{3^n-8}$ was an alternating series, and assuming $\sum_{n=3}^{\infty} \frac{1}{3^n-8}$ was a geometric series. Also, many people did not realize that $\frac{1}{3^n-8}$ was greater than not less than $\frac{1}{3^n}$ for $n \geq 3$.

(d) $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}$

This one can either be done using the ratio test or the alternating series test. To apply the ratio test, consider

$$a_{n+1} = (-1)^{n+1} \frac{(n+1)!}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)(2(n+1)-1)}$$

and

$$a_n = (-1)^n \frac{n!}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}.$$

Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| (-1)^{n+1} \frac{(n+1)!}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)(2(n+1)-1)} \cdot \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{(-1)^n\, n!} \right|$$
$$= \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1.$$

It follows $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}$ converges absolutely, hence it converges.

A common mistake was to write the denominator of the fraction $\frac{n!}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}$ as $(2n-1)!$, which it is not! It is only the product of **odd** integers up and including $2n-1$. This led to an incorrect computation of the limit. Some other common mistakes were to use the alternating series test but either forget to check that the absolute values of the terms were decreasing or to check that the limit of the terms was zero as $n \to \infty$. Many people lost one point by forgetting to provide at least some justification as to why the limit of the terms was zero.

4. (a) The center of the series is $x = 2$.

   (b) Let's think about the radius of convergence R. The fact that the series converges at $x = 4$ means that $R \geq 2$, the fact that the series diverges at $x = 0$ means that $R \leq 2$. So we know that $R = 2$. What happens at the endpoints? The endpoints are 0 and 4, and we know what happens there already. Thus the interval of convergence is $(0, 4]$.

   (c) i. $\frac{(-1)^n (2)^n}{b_n} = \frac{(-2)^n}{b_n} = \frac{(0-2)^n}{b_n}$
   So this series is just the power series evaluated at $x = 0$, and we're told that this diverges.

   ii. Similarly, this is the power series evaluated at $x = 3$, and 3 is in the interval of convergence. So this series must converge.

   iii. If the previous series converges then we must have that $\lim_{n\to\infty} \frac{1}{b_n} = 0$. So we can't possibly have $\lim_{n\to\infty} b_n = 0$. So this series must diverge by the n^{th} term test.

5. (a) (7 points) We use a geometric series to find a power series representation of $\frac{1}{8x^3+1}$:

$$\frac{1}{8x^3+1} = \frac{1}{1-(-8x^3)} = \sum_{n=0}^{\infty}(-8x^3)^n = \sum_{n=0}^{\infty}(-1)^n 8^n x^{3n}.$$

(this is only true for certain x; see below). So

$$\frac{x}{8x^3+1} = x\sum_{n=0}^{\infty}(-1)^n 8^n x^{3n} = \sum_{n=0}^{\infty}(-1)^n 8^n x^{3n+1}.$$

Very common mistakes were forgetting to distribute the power to the factor of 8, or miscalculating the power of x.

   (b) (3 points) The quickest way to solve this problem is to note that the interval of convergence of *any* geometric series $\sum_{n=0}^{\infty} ar^n$ is $|r| < 1$; in our case $r = -8x^3$ so the series $\sum_{n=0}^{\infty}(-8x^3)^n$ converges if and only if

$$\left|-8x^3\right| < 1 \implies |x| < \sqrt[3]{\frac{1}{8}} = \frac{1}{2}.$$

Multiplying a series by x doesn't change its interval of convergence.
If you do the ratio test on the series from (a), you get a ratio of successive terms equal to

$$\left|\frac{(-1)^{n+1}8^{n+1}x^{3(n+1)+1}}{(-1)^n 8^n x^{3n+1}}\right| = \left|8x^3\right|,$$

which tends to $|8x^3|$ as $n \to \infty$. If this limit is less than 1, the series converges, so the radius of convergence is $\frac{1}{2}$. However, here it is also necessary to test the endpoints of the interval. If $x = \frac{1}{2}$ the series is

$$\sum_{n=0}^{\infty}(-1)^n 8^n \left(\frac{1}{2}\right)^{3n} \cdot \frac{1}{2} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \cdots,$$

which cannot converge by the nth term test for divergence. The same is true of the series evaluated at $x = -\frac{1}{2}$:

$$\frac{1}{2}\sum_{n=0}^{\infty} 1^n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots,$$

The radius of convergence was worth one point as was each endpoint. We read along with the student; if the power series in (a) was wrong but its interval of convergence was computed correctly in (b) the only deductions were from (a).

6. (a) (3 points) We know the Maclaurin series for e^u is

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

so

$$
\begin{aligned}
e^{-x^3} &= \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n e^{3n}}{n!} \\
xe^{-x^3} &= \sum_{n=0}^{\infty} \frac{(-1)^n e^{3n+1}}{n!}
\end{aligned}
$$

(b) (3 points) We need only integrate the series from (a) term-by-term. We get

$$\int xe^{-x^3}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n e^{3n+2}}{(3n+2)n!}$$

The constant term is necessary because this question asks for an indefinite integral. Neglecting it resulted in a $\frac{1}{2}$-point deduction.

(c) (5 points) When evaluating the power series above at $x = 0.1$, we have the series

$$s = \sum_{n=0}^{\infty} \frac{(-1)^n}{10^{3n+2}(3n+2)n!},$$

which we can estimate with the Alternating Series Estimation Theorem. Notice that if b_n is the absolute value of the nth term of the above series, we have

$$
\begin{aligned}
b_0 &= \frac{1}{10^2 \cdot 2 \cdot 1} = \frac{1}{200} \\
b_1 &= \frac{1}{10^5 \cdot 5 \cdot 1} = \frac{1}{500,000} \\
b_2 &= \frac{1}{10^8 \cdot 8 \cdot 2} = \frac{1}{16 \times 10^8} < \frac{1}{10^9}.
\end{aligned}
$$

By the A.S.E.T. we have

$$|s - s_1| < b_2 < 10^{-9},$$

so a good-enough approximation is

$$s_1 = b_0 - b_1 = \frac{1}{200} - \frac{1}{500,000}.$$

A consequence of the hypotheses of the A.S.T. is that if s_n ends with a positive term, it is more than the sum of the series, and if s_n ends with a negative term, the estimate is too small. Thus s_1 is slightly less than s.

Any partial sum beyond s_1 was also accepted, as long as there was proper justification that the first neglected term was small enough, and the too small/too large question was answered accordingly.

A common error was forgetting to sum from $n = 0$, and estimating the sum of the first n terms with the nth term (not the first *neglected* term.)

One point was given for the estimate itself, three for the argument that it was close enough, and one for the too small/too large question. We read along with the student as much as possible.

7. (a) (4 points)

Statement (i) is false. The harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by the integral test) even though $\lim_{n\to\infty} \frac{1}{n} = 0$. Statement (ii) is true. (The condition $\lim_{n\to\infty} a_n = 0$ is necessary (but not sufficient) for convergence.

(b) (4 points)

Answer: (iv). Sayid's aims are twofold. First, he must expand the series about a number for which the numerical value of the cube root is known. This eliminates choice (iii). Second: he would like quick convergence, so he would like to evaluate as close as possible to the point about which he is expanding.

Practically speaking, we would estimate that the cube root of 7.4 is a bit less than two, because we know that the cube root of 8 is 2. Therefore we should expand around 8 and evaluate at 7.4.

In fact, choice (i) can be eliminated because we can't even write the Taylor series about $x = 0$. The function isn't differentiable there. Similarly, choice (ii) can be eliminated because the series expansion for $x^{1/3}$ won't converge at 6.4. However, it was not necessary to realize that in order to see that choice (iv) is better!

(c) (4 points) An infinite geometric series $\sum_{n=0}^{\infty} a\,r^n$ converges to $\frac{a}{1-r}$ provided $|r| < 1$.

We want $\frac{a}{1-r} = 3$ or $a = 3(1-r)$. Choose some value of r, with the only condition being $|r| < 1$. Say $r = \frac{1}{3}$. Then $a = 3(1 - \frac{1}{3}) = 3(\frac{2}{3}) = 2$.

So the series $\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^n = 3$.

(d) (7 points)

i. (2 points)

For $n = 0$ we have $\cos 0 \frac{(x-3)^0}{0!} = 1 \cdot 1 = 1$.

For $n = 1$ we have $\cos(\pi/2)\frac{(x-3)^1}{1!} = 0 \cdot (x-3) = 0$.

For $n = 2$ we have $\cos(\pi)\frac{(x-3)^2}{2!} = -1 \cdot \frac{(x-3)^2}{2!} = -\frac{(x-3)^2}{2!}$.

For $n = 3$ we have $\cos(3\pi/2)\frac{(x-3)^3}{3!} = 0 \cdot \frac{(x-3)^3}{3!} = 0$.

For $n = 4$ we have $\cos(2\pi)\frac{(x-3)^4}{4!} = 1 \cdot \frac{(x-3)^4}{4!} = \frac{(x-3)^4}{4!}$.

$$1 - \frac{(x-3)^2}{2} + \frac{(x-3)^4}{4!} + \ldots$$

ii. (1 point) 1. The value at 3 is simply $f(3) = 1$ (All terms after the first are zero at $x = 3$.)

iii. (4 points) The function f has a local maximum at $x = 3$. The first derivative is zero and the second derivative is negative. (Alternatively, realize that this function is $\cos x$ shifted over 3: $\cos(x-3)$, so it has a local maximum at $x = 3$.