Harvard University
Math 1b. Second Semester Calculus
First Exam

Name Answer Key (APopa)

October 28, 1998

First Exam

- Do not open this exam booklet until you are directed to do so.
- You have 120 minutes to earn 80 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any problem. Read them all through first and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.
- Good luck!

Problem	Points	Grade
1	15	15
2	10	10
3	10	10
4	10	10
5	10	10
6	15	15
7	10	10
Total	80	80

Please circle your section:

MWF 10:00	MWF 10:00	MWF 10:00	MWF 11:00	MWF 11:00	MWF 12:00
Brian Conrad	Andy Engelward	Eric Towne	Noam Elkies	Robert Pollack	Andy Engelward

TTh 10:00	TTh 10:00	TTh 10:00	TTh 11:30
Tomas Klenke	Joel Rosenberg	Eric Towne	Heather Russell

1. (15 pts)

1a. Determine whether each of the following series converges or diverges. Justify your answers.

(i) $\sum_{k=1}^{\infty}\left(1-\frac{1}{k^2}\right)$

Since $\lim_{k\to\infty}\left(1-\frac{1}{k^2}\right)=1\neq 0$, the series <u>diverges</u> by the N^{th} term test for divergence.

(ii) $\sum_{n=0}^{\infty}\frac{n^2+1}{n!}$

RATIO TEST $\lim_{n\to\infty}\frac{\frac{(n+1)^2+1}{(n+1)!}}{\frac{n^2+1}{n!}}=\lim_{n\to\infty}\frac{1}{n+1}\cdot\frac{(n+1)^2+1}{n^2+1}=0<1$

so the series <u>CONVERGES</u>

(iii) $\sum_{k=1}^{\infty}\frac{\cos^2 k}{k^2+1}$

Since $\frac{\cos^2 k}{k^2+1}<\frac{1}{k^2+1}<\frac{1}{k^2}$, the series converges by comparison with the p-series $\sum_{k=1}^{\infty}\frac{1}{k^2}$. This p-series has $p>1$, so it converges.

1b. For what real values of a do the following series converge? Explain your answers.

(i) $\sum_{k=1}^{\infty} a$

The sequence $a, a, a, a, \ldots$ converges to a, so the series diverges if $a \neq 0$ by the nth term test for divergence.
If $a = 0$, the series converges.

(ii) $\sum_{k=1}^{\infty} (-1)^k k^a$

If $a > 0$, $\lim_{k \to \infty} k^a = \infty$, so the series diverges by the Nth term test for divergence.

If $a = 0$, $\sum_{k=1}^{\infty} (-1)^k = -1 + 1 - 1 + 1 - \ldots$ diverges.

If $a < 0$, then $\lim_{k \to \infty} k^a = 0$ and $1 > 2^a > 3^a > 4^a > \ldots$ (the terms are decreasing in magnitude)

so

Conclusion: The series converges for $\boxed{a < 0}$ only.

2. (10 pts) For each power series below, give the radius and interval of convergence. Be sure to check convergence at the endpoints.

a. $\displaystyle\sum_{k=0}^{\infty} \frac{k!(x-3)^k}{2^k}$

$$\lim_{k\to\infty} \frac{\frac{(k+1)!\,|x-3|^{k+1}}{2^{k+1}}}{\frac{k!\,|x-3|^k}{2^k}} = \lim_{k\to\infty} \frac{k+1}{2}\cdot|x-3| = \infty, \text{ for } x \neq 3.$$

So $\underbrace{\text{the radius}}_{\text{by the ratio test}}$ of convergence is $R=0$, the series converges only for $x=0$.

b. $\displaystyle\sum_{k=1}^{\infty} \frac{(x-4)^k}{k\cdot 3^k}$

$$\lim_{k\to\infty} \frac{\frac{|x-4|^{k+1}}{(k+1)3^{k+1}}}{\frac{|x-4|^k}{k\cdot 3^k}} = \lim_{k\to\infty} \frac{k}{k+1}\cdot\frac{|x-4|}{3} = \frac{|x-4|}{3}$$

By the ratio test, the series $\begin{cases} \text{converges for } \left|\frac{x-4}{3}\right|<1, \ |x-4|<3 \\ \text{diverges for } \left|\frac{x-4}{3}\right|>1, \ |x-4|>3. \end{cases}$

So $R=3$ is the radius of convergence.

+1 4 7

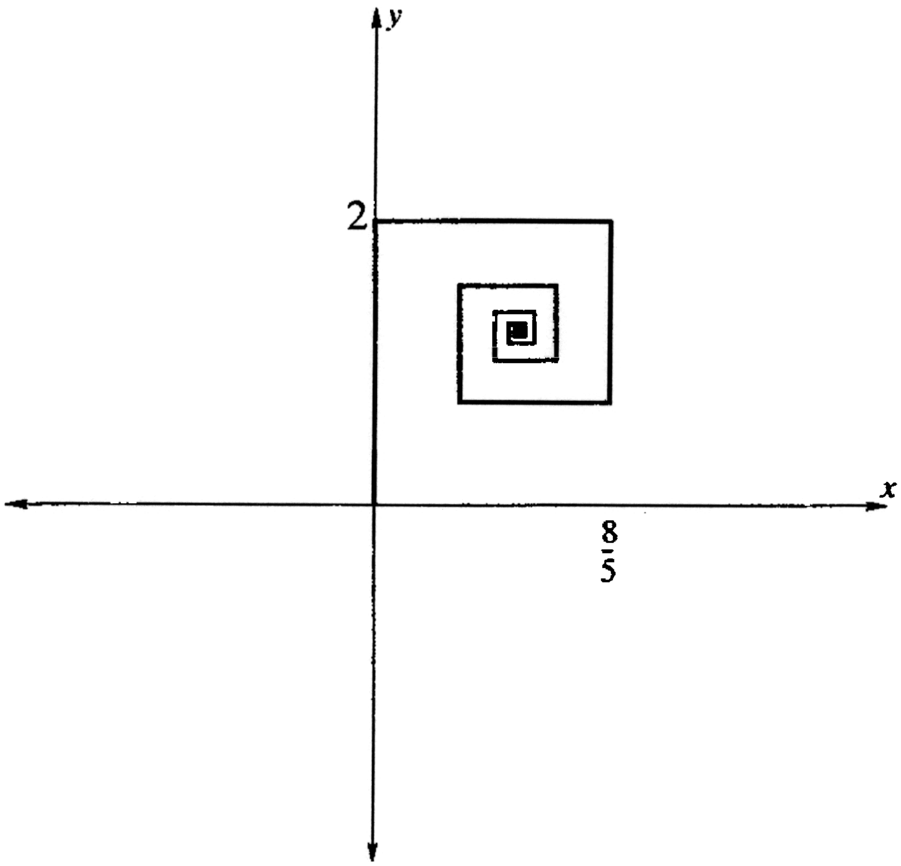
Endpoints: $x=+1$: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges (alternating harmonic series)

$x=7$ $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k\,3^k} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (harmonic)

So the interval of convergence is $[1,7)$.

3. (10 pts) I want to draw a spiral, but since I am not that good at drawing curves, I've decided to draw a "spiral with corners" instead, as follows:

I start at the origin $(0,0)$ of the xy-plane, and draw a segment 2 units long, vertically upwards. Then I turn right by 90°, and draw a segment $\frac{4}{5}$ as long as the first segment. Then I make another 90° right turn, and draw a segment $\frac{4}{5}$ as long as the second segment. I continue this way, and get my spiral:



(Question continues on next page)

a. What is the total distance my pen will eventually travel?

TOTAL DISTANCE: $L = 2 + 2\cdot\frac{4}{5} + 2\cdot\left(\frac{4}{5}\right)^2 + 2\cdot\left(\frac{4}{5}\right)^3 + \cdots = 2\cdot\frac{1}{1-\frac{4}{5}} = 10$

(geometric series)

b. Find the coordinates (x_0, y_0) of the point P in the xy-plane at which I will eventually end up. (Hint: consider the x-coordinate after each segment is drawn, and then consider the y-coordinate after each segment is drawn.)

$$y_0 = 2 - 2\cdot\left(\frac{4}{5}\right)^2 + 2\cdot\left(\frac{4}{5}\right)^4 - 2\cdot\left(\frac{4}{5}\right)^6 + \cdots\cdots$$

$$= 2\left[1 + \left(-\frac{16}{25}\right) + \left(\frac{16}{25}\right)^2 - \left(\frac{16}{25}\right)^3 + \cdots\right]$$

Geometric series with $r = -\frac{16}{25}$:

$$= 2\cdot\frac{1}{1+\frac{16}{25}} = \frac{50}{41}$$

$$X_0 = 2\cdot\frac{4}{5} - 2\left(\frac{4}{5}\right)^3 + 2\left(\frac{4}{5}\right)^5 - 2\left(\frac{4}{5}\right)^7 + \cdots$$

$$= \frac{8}{5}\left(1 - \frac{16}{25} + \left(\frac{16}{25}\right)^2 - \left(\frac{16}{25}\right)^3 + \cdots\right) = \frac{8}{5}\cdot\frac{1}{1+\frac{16}{25}} = \frac{40}{41}$$

So $(X_0, y_0) = \left(\frac{40}{41}, \frac{50}{41}\right)$

4. (10 pts)

a. Find the second degree Taylor polynomial for $f(x) = \sqrt{x}$ centered at $x = 4$.

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$ $\qquad f''(x) = -\frac{1}{4} x^{-3/2}$

$f(4) = \sqrt{4} = 2,\ f'(4) = \frac{1}{2} \cdot \frac{1}{\sqrt{4}} = \frac{1}{4},\ f''(4) = -\frac{1}{4} \cdot \frac{1}{(\sqrt{4})^3} = -\frac{1}{32}$

$P_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$

b. Use your answer to **a** to give an approximation to $\sqrt{3.9}$. (You need not simplify your answer.)

$\sqrt{3.9} \simeq p_2(3.9) = 2 - \frac{1}{4} \cdot (.1) - \frac{1}{64} \cdot (.1)^2$

c. If you wanted to approximate $\sqrt[3]{31}$ by using a Taylor series for $\sqrt[3]{x}$, where would you center your Taylor polynomial? Explain your reasoning briefly. Do not attempt to approximate $\sqrt[3]{31}$; just say how you would do it.

I would center the Taylor polynomial at $x=27$, because $f(27), f'(27), f''(27)\ldots$ are all easily computable ratios of integers

$f(27) = \sqrt[3]{27} = 3$

$f'(27) = \frac{1}{3} x^{-\frac{2}{3}}\Big|_{x=27} = \frac{1}{3} \cdot \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{27}$ etc.

Thus the Taylor polynomial about $x=27$ has rational coefficients

5. (10 pts) Show that if we try to compute $\sqrt[3]{1.06}$ by approximating the function

$$g(x) = \sqrt[3]{1+x}$$

by the polynomial $1 + \frac{1}{3}x$, then the error will be less than .0005.

To compute this, you can use the binomial theorem with $p = 1/3$

$$(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3 + \cdots$$

Alternatively - work out the Taylor series about $x=0$ from scratch

$$g(x) = \sqrt[3]{1+x} = (1+x)^{1/3}$$

so $g'(x) = \frac{1}{3}(1+x)^{-2/3}$ and $g(0) = 1$

$g''(x) = -\frac{2}{9}(1+x)^{-5/3}$ $\quad g'(0) = \frac{1}{3}$

$g'''(x) = \frac{10}{27}(1+x)^{-8/3}$ $\quad g''(0) = -\frac{2}{9}$

$g'''(0) = \frac{10}{27}$

⋮

from here on, the signs alternate

Maclaurin series for $\sqrt[3]{1+x}$ is $1 + \frac{1}{3}x - \left(\frac{\frac{2}{9}}{2!}\right)x^2 + \left(\frac{\frac{10}{27}}{3!}\right)x^3 \ldots$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 \ldots$$

it's an alternating series after first two terms for $x > 0$, (with decreasing terms if $x = .06$, where it converges)

when $x = .06$ then $g(.06) = \sqrt[3]{1+.06} = \sqrt[3]{1.06}$

$$\approx 1 + \frac{1}{3}(.06), \text{ with}$$

$$|\text{error}| < \frac{1}{9}(.06)^2 = \frac{1}{9}\cdot\left(\frac{6}{100}\right)^2$$

$$= \frac{36}{9}\cdot\frac{1}{10{,}000} = .0004$$

$$< .0005$$

6. (15 pts) Let $f(x)$ be given by the power series

$$f(x) = \sum_{k=1}^{\infty} \frac{(x-2)^k}{\sqrt{k}}$$

a. Determine the radius of convergence and the interval of convergence of this series. Be sure to check convergence at the endpoints.

$$\lim_{k\to\infty} \frac{\frac{|x-2|^{k+1}}{\sqrt{k+1}}}{\frac{|x-2|^k}{\sqrt{k}}} = \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k+1}} \cdot |x-2| = |x-2|.$$

By the ratio test, the series converges for $|x-2|<1$, diverges for $|x-2|>1$.

So the radius of convergence is 1. $|x-2|<1$ means, $1<x<3$

Endpoints: $x=1$: $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converges (alternating series test, since $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$, and $\frac{1}{\sqrt{k}} > \frac{1}{\sqrt{k+1}}$)

$x=3$: $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges (p-series, $p=\frac{1}{2}$)

So the interval of convergence is $[1, 3)$

b. Write out the first four nonzero terms of the power series for the derivative $\frac{df}{dx}$.

What are the radius and interval of convergence of the power series for $\frac{df}{dx}$?

$$\frac{df}{dx} = \sum_{k=1}^{\infty} \frac{k(x-2)^{k-1}}{\sqrt{k}} = \sum_{k=1}^{\infty} \sqrt{k}(x-2)^{k-1} = 1 + \sqrt{2}(x-2) + \sqrt{3}(x-2)^2 + \sqrt{4}(x-2)^3 + \cdots$$

The radius of convergence is the same as that for $f(x)$, that is $R=1$

Endpoints: $x=1$: $\sum_{k=1}^{\infty} (-1)^k \sqrt{k}$ diverges by the divergence test ($\lim_{k\to\infty} (-1)^k \sqrt{k} \neq 0$.)

$x=3$: $\sum_{k=1}^{\infty} \sqrt{k}$ diverges by the divergence test ($\lim_{k\to\infty} \sqrt{k} \neq 0$).

So the interval ~~of~~ convergence is $(1, 3)$

c. Let $g(x)$ be an antiderivative for $f(x)$, i.e. $\frac{dg}{dx} = f(x)$, and such that $g(\cancel{0}\,2) = 0$. What are the first four nonzero terms of the power series for $g(x)$?
What are the radius and interval of convergence of the series for $g(x)$?

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \cdot \frac{1}{(k+1)} (x-2)^{k+1} + C$$ (integrating term by term)

$\cancel{x} = 2$: $C = g(2) = 0$

So $g(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \cdot \frac{(x-2)^{k+1}}{k+1}$

radius of convergence is the same $R = 1$.

Endpoints: $x = 1$: $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}(k+1)}$ converges (alternating series test, $\lim_{k\to\infty} \frac{1}{\sqrt{k}(k+1)} = 0$, and $\frac{1}{\sqrt{k}(k+1)} > \frac{1}{\sqrt{k+1}((k+1)+1)}$

$x = 3$ $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} \cdot (k+1)}$ converges by limit comparison

with the p-series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ $\left(\lim_{k\to\infty} \left(\frac{1}{\sqrt{k}(k+1)} \Big/ \frac{1}{k^{3/2}}\right) = 1\right)$

So the interval of convergence is $[1, 3]$

7. (10 pts)

a. Give the first four nonzero terms of the Taylor series at $x = 0$ for the function

$$f(x) = \frac{e^{x^2} + e^{-x^2}}{2}.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} \cancel{+} \frac{x^6}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}$$

$$\frac{e^{x} + e^{-x^2}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \frac{(1+(-1)^k)}{2} = 1 + \frac{x^4}{2!} + \frac{x^8}{4!} + \frac{x^{12}}{6!} +$$

b. Find the Maclaurin series for $x - \sin x$. (Write out the first four nonzero terms.)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$x - \sin x = \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots$$

c. Using your answer to b, compute

$$\lim_{x\to 0} \frac{x - \sin x}{x^3}.$$

by part (b)

$$\frac{x - \sin x}{x^3} = \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^5}{7!}$$

So $\lim_{x\to 0} \frac{x-\sin x}{x^3} = \frac{1}{3!} = \frac{1}{6}$ (the series on the right hand side evaluated at $x=0$).