Math 1a. Lecture 4
Additional Techniques of Integration (Trigonometric Integrals and Trigonometric Substitution)

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1 Goals

- To be able to evaluate trigonometric integrals.
- To be able to evaluate definite and indefinite integrals using trigonometric substitution.

2 Trigonometric Integrals

- If the integrand contains an odd power, it can usually be reduced to a substitution problem if we use the identity, $\cos^2 x + \sin^2 x = 1$. For example,
$$\int \sin^3 x\,dx = \int (1 - \cos^2 x) \sin x\,dx$$
- If the integrand contains only even powers such as
$$\int \cos^2 x\,dx,$$
we can use the half-angle formulas,
$$\begin{aligned} \sin^2 x &= \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x &= \frac{1}{2}(1 + \cos 2x). \end{aligned}$$
These follow from the double angle formula for $\cos x$.

3 Trigonometric Substitution

Integrals containing an expression of the form $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, or $\sqrt{a^2 + x^2}$ can be evaluated using trigonometric substitution.[1] Consider the following example,

$$\int \frac{x^3}{\sqrt{1-x^2}}\, dx,$$

where $|x| < 1$. We can make the substitution

$$\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta\, d\theta. \end{aligned}$$

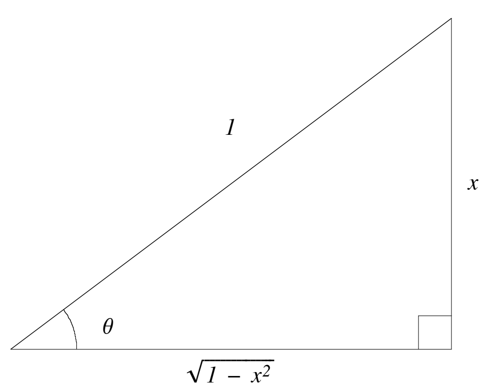
Then

$$\begin{aligned} \int \frac{x^3}{\sqrt{1-x^2}}\, dx &= \int \frac{\sin^3\theta\cos\theta}{\sqrt{1-\sin^2\theta}}\, d\theta \\ &= \int \sin^3\theta\, d\theta \\ &= \frac{1}{3}\cos^3\theta - \cos\theta + C. \end{aligned}$$

Since $\cos\theta = \sqrt{1-x^2}$,

$$\int \frac{x^3}{\sqrt{1-x^2}}\, dx = \frac{1}{3}(1-x^2)^{3/2} - \sqrt{1-x^2} + C.$$

Use the picture below.



[1]Stewart's explanation is lacking here.

4 Worksheet Exercises

Evaluate each of the following integrals

1. $\int \cos^3 x \sin^4 x \, dx$

2. $\int \sin^2 x \cos^2 x \, dx$

3. $\int \sec^3 x \tan^3 x \, dx.$

4. $\int \sec\theta \, d\theta = \int \frac{\cos\theta}{\cos^2\theta} \, d\theta = \int \frac{\cos\theta}{1 - \sin^2\theta} \, d\theta.$ *Hint*: Make use of the identity

$$\frac{2}{1-u^2} = \frac{1}{1-u} + \frac{1}{1+u}.$$

5. $\int \sqrt{4 - x^2} \, dx$

6. $\int \frac{1}{\sqrt{4 + x^2}} \, dx$

References

- §5.7 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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