Math 1a. Lecture 6
Approximate Integration

T. Judson

Spring 2006

1 Goals[1]

- To understand and be able to the Midpoint Rule to approximate definite integrals.
- To understand and be able to the Trapezoid Rule to approximate definite integrals.
- To understand and be able to Simpson's Rule to approximate definite integrals.
- To understand and be able to apply error estimates for the Midpoint, Trapezoid, and Simpson's Rule

2 Approximating Integrals Numerically

Suppose that we wish to compute

$$\int_0^1 \sin(x^2)\, dx.$$

Although the Fundamental Theorem of Calculus guarantees that

$$F(x) = \int_0^t \sin(t^2)\, dt$$

is an antiderivative of $\sin(x^2)$, it is not possible to express this antiderivative in terms of elementary functions.

[1] A calculator is useful for this lesson.

3 Midpoint and Trapezoid Rules

We can certainly use left and right sums to estimate

$$\int_a^b f(x)\,dx.$$

That is,

$$\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$
$$\int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta = (b-a)/n$. A better strategy is to estimate the height of the function in each subinterval $[x_{i-1}, x_i]$ using the midpoint of the subinterval, $(x_{i-1}+x_i)/2$. This gives us the *Midpoint Rule*,[2]

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x.$$

If we approximate $f(x)$ by a linear function on each subinterval $[x_{i-1}, x_i]$, then we have the *Trapezoid Rule*,

$$\int_a^b f(x)\,dx \approx T_n = \sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2}\Delta x.$$

4 Example

Consider $I = \int_0^1 \sin(x^2)\,dx$.

1. If $n = 8$, use a Left Sum to estimate I.
2. If $n = 8$, use a Right Sum to estimate I.
3. If $n = 8$, use the Midpoint Rule to estimate I.
4. If $n = 8$, use the Trapezoid Rule to estimate I.

[2]It is easy to develop the Midpoint and Trapezoid Rules using the graph of the function. It is also important to write out each of these rules *without* using summation notation.

5 Simpson's Rule

Simpson's Rule is based on using quadratic functions to estimate $f(x)$, instead of constant or linear functions.

$$\int_a^b f(x)\,dx \approx S_n = [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]\frac{\Delta x}{3},$$

where $\Delta = (b-a)/n$.

Understanding Simpson's Rule geometrically is a three-step process.[3]

1. Partition $[a, b]$ into an *even* number of equal subdivisions, each with width $\Delta = (b-a)/n$. Consider successive *pairs* of subdivisions.

2. Each pair involves *three* equally spaced points: $-h$, 0, and h. Over each such double subinterval, replace the integrand f with a quadratic that passes through the points

$$(-h, f(-h))$$
$$(0, f(0))$$
$$(h, f(h)).$$

   For algebraic reasons, only one such function q exists. If the three points are collinear, then q will be a linear function.

3. The area under (over if $f < 0$) each small parabolic arc is

$$\begin{aligned}\int_{-h}^{h} Ax^2 + Bx + C\,dx &= \underbrace{\int_{-h}^{h} Ax^2 + C\,dx}_{\text{even}} + \underbrace{\int_{-h}^{h} Bx\,dx}_{\text{odd}} \\ &= 2\int_0^h Ax^2 + C\,dx \\ &= \frac{h}{3}(2Ah^2 + 6C).\end{aligned}$$

[3]Leave this section for last in case there is not enough time. Students can read about the exact steps on pp. 416–417 in Stewart.

Since the quadratic passes through the points $P_0 = (-h, y_0)$, $P_1 = (0, y_1)$, and $P_2 = (h, y_2)$, we know that

$$\begin{aligned} y_0 &= Ah^2 - Bh + C \\ y_1 &= C \\ y_2 &= Ah^2 + Bh + C \end{aligned}$$

Thus,

$$y_0 + 4y_1 + y_2 = 2Ah^2 + 6C$$

Summing up all $n/2$ contributions give S_n, Simpson's Rule.

6 Example

Use Simpson's rule to estimate $I = \int_0^1 \sin(x^2)\, dx$ if $n = 8$.

7 Error for the Midpoint and Trapezoid Rules

How accurate is each method. If we examine graphs approximated by linear functions or constant functions, it is clear that concavity of the function will determine error. Suppose that concavity is bounded, say $|f''(x)| \leq K$ on the interval $[a, b]$. Among all functions that satisfy this inequality, the quadratic

$$q(x) = \frac{Kx^2}{2}$$

is the worst offender since $|q''(x)| \leq K$ for all x in the interval $[a, b]$. How much error can happen over a subinterval. To simplify matters, we take $[0, h]$ to be our subinterval. Then

$$I_h = \int_0^h q(x)\, dx = \int_0^h \frac{Kx^2}{2}\, dx = \frac{Kh^3}{6}$$

$$M_h = q\left(\frac{h}{2}\right) \cdot h = \frac{Kh^3}{8}$$

$$T_h = \frac{q(0) + q(h)}{2} \cdot h = \frac{Kh^3}{4}$$

The errors can now be computed to be

$$\begin{aligned} |I_h - M_h| &= \frac{Kh^3}{24} \\ |I_h - T_h| &= \frac{Kh^3}{12}. \end{aligned}$$

Now replace h by $(b-a)/n$ to get the following error estimates.

> Let $|f''(x)| \le K$ on the interval $[a,b]$. If E_M and E_T are the errors in the Trapezoid and Midpoint Rules, then
>
> $$\begin{aligned} |E_T| &\le \frac{K(b-a)^3}{12n^2} \\ |E_M| &\le \frac{K(b-a)^3}{24n^2}. \end{aligned}$$

Let $I = \int_0^1 \sin(x^2)\,dx$.

1. How many subdivisions are needed to assure that the error is less than 10^{-6} if we use the Midpoint Rule to estimate I?

2. How many subdivisions are needed to assure that the error is less than 10^{-6} if we use the Trapezoid Rule to estimate I?

8 Error for Simpson's Rule

> Suppose that $|f^{(4)}(x)| \le K$ on the interval $[a,b]$. If E_S is the errors in Simpson's Rule, then
>
> $$|E_S| \le \frac{K(b-a)^5}{180n^4}.$$
>
> If we use Simpson's Rule to estimate $I = \int_0^1 \sin(x^2)\,dx$, how many subdivisions are needed to assure that the error is less than 10^{-6}?

References

- §5.9 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

February 17, 2006