Math 1a. Lecture 7
Improper Integrals

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1 Goals

- To understand and be able to evaluate integrals of the form $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^a f(x)\,dx$, and $\int_{-\infty}^\infty f(x)\,dx$ (Improper Integrals of Type 1).
- To understand and be able to evaluate integrals of the form $\int_a^b f(x)\,dx$, where f is discontinuous at a or b (Improper Integrals of Type 2).
- To understand and be able to apply the comparison test for improper integrals.

2 Improper Integrals of Type 1

- Evaluate $\displaystyle\int_1^t e^{-x}\,dx$. Does this integral make sense for all t? What if $t \to \infty$?
- Evaluate $\displaystyle\int_1^t \frac{1}{x}\,dx$. Does this integral make sense for all t? What if $t \to \infty$?

Integrals of Type 1 converge to the following limits. If the limit does not exist, then the integral diverges.

- $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$
- $\displaystyle\int_{-\infty}^a f(x)\,dx = \lim_{t\to\infty}\int_t^a f(x)\,dx$

- $\displaystyle\int_{\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$, provided both of the integrals on the right-hand side converge. We may use any value for a.

3 Improper Integrals of Type 2

Integrals of Type 1 converge to the following limits. If the limit does not exist, then the integral diverges.

- If f is continuous on $[a, b)$ and discontinuous at b, then

$$\int_{a}^{b} f(x)\,dx = \lim_{t\to b} \int_{a}^{t} f(x)\,dx.$$

- If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_{a}^{b} f(x)\,dx = \lim_{t\to a} \int_{t}^{b} f(x)\,dx.$$

4 The Comparison Test for Improper Integrals

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then

- If $\displaystyle\int_{a}^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\int_{a}^{\infty} g(x)\,dx$ is convergent.
- If $\displaystyle\int_{a}^{\infty} g(x)\,dx$ is divergent, then $\displaystyle\int_{a}^{\infty} f(x)\,dx$ is divergent.

5 Worksheet Problems

1. Evaluate $\displaystyle\int_{1}^{t} e^{-x}\,dx$. Does this integral make sense for all t? What if $t \to \infty$?
2. Evaluate $\displaystyle\int_{1}^{t} \frac{1}{x}\,dx$. Does this integral make sense for all t? What if $t \to \infty$?
3. Does $\displaystyle\int_{1}^{\infty} \frac{1}{x^2}\,dx$ converge? If so, to what?

4. Find all values of p such that the integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges.

5. Does $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$ converge? If so, to what?

6. Does $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$ converge? If so, to what?

7. Does $\int_1^\infty \frac{1}{x^5+1}\,dx$ converge? Why or why not?

8. Does $\int_0^\infty \frac{1}{e^x+x}\,dx$ converge? Why or why not?

9. Does $\int_1^\infty \frac{\sin x}{x^2}\,dx$ converge? Why or why not?

10. Find all values of p such that the integral $\int_1^e \frac{1}{x(\ln)^p}\,dx$ converges.

References

- §5.10 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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