Math 1a. Lecture 11
Arc Length and Average Value of a Function

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1 Goals

- To understand and be able to compute arc length.
- To understand and be able to compute the average value of a function.

2 Arc Length

We can approximate the length of a curve C be calculating the length of inscribed polygonal arc, C_n. Let C be given by the function $y = f(x)$ on the interval $[a, b]$ and consider a subdivision

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The length of the ith line segment in our polygonal arc is

$$\sqrt{[f(x_i) - f(x_{i-1})]^2 + (x_i - x_{i-1})^2}.$$

Thus, the length of the entire polygonal arc is

$$\begin{aligned} |C_n| &= \sum_{i=1}^{n} \sqrt{[f(x_i) - f(x_{i-1})]^2 + (x_i - x_{i-1})^2} \\ &= \sum_{i=1}^{n} \sqrt{(\Delta y_i)^2 + (\Delta x_i)^2} \\ &= \sum_{i=1}^{n} \left(\sqrt{\left(\frac{\Delta y_i}{\Delta x_i} \right)^2 + 1} \right) \cdot \Delta x_i \end{aligned}$$

Therefore, the length of the arc can be found by letting $n \to \infty$

$$|C| = \int_a^b \sqrt{[f'(x)]^2 + 1}\, dx.$$

3 Example

The length of $y = \sin x$ from $x = 0$ to $x = \pi$ is

$$\int_0^\pi \sqrt{\cos^2 x + 1}\, dx \approx 3.8202.$$

Most integrals coming from arc length problems cannot be evaluated in terms of elementary functions.

4 Finding the Average Temperature

1. How would you find the average temperature in Boston over a 24-hour period?
2. How would you calculate the average temperature in Boston given the following data?

Time	Temperature
2 PM	68 °F
6 PM	68
10 PM	58
2 AM	55
6 AM	54
10 AM	65
2 PM	74

3. How would you calculate the average temperature in Boston given that the temperature is given by

$$f(t) = \frac{1}{9}t^2 - \frac{29}{12}t + 68,$$

where t is given in hours since 2 PM and $f(t)$ is given in degrees Fahrenheit?

(a) Partition the interval $[0, 24]$ into n equal subintervals, so that the length of each subinterval is

$$\Delta t = \frac{24}{n}$$

and

$$t_i = 0 + i\Delta t.$$

(b) Then the average of the temperatures taken at times t_1, t_2, $\ldots$, t_n is

$$\begin{aligned}\frac{f(t_1) + f(t_2) + \cdots + f(t_n)}{n} &= \frac{f(t_1) + f(t_2) + \cdots + f(t_n)}{24/\Delta t} \\ &= \frac{(f(t_1) + f(t_2) + \cdots + f(t_n))\,\Delta t}{24} \\ &= \frac{f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t}{24} \\ &= \frac{1}{24}\sum_{i=1}^{n} f(t_i)\Delta t.\end{aligned}$$

(c) As $n \to \infty$, the average temperature converges to

$$\frac{1}{24}\int_0^{24} f(t)\,dt.$$

(d) In this case,

$$\frac{1}{24}\int_0^{24} f(t)\,dt = \frac{181}{3} = 60\frac{1}{3}.$$

5 Average Value of a Function

Define the *average value of a function* f on the interval $[a, b]$ to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

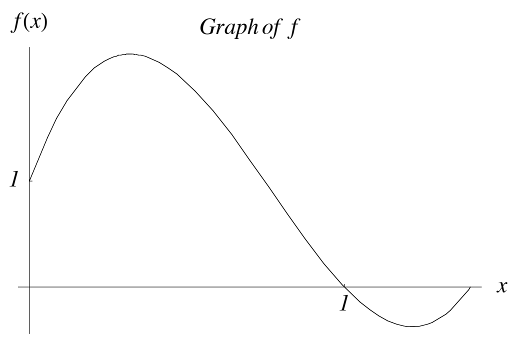
Note that if we let c be the average value of f on the interval $[a, b]$, then

$$c(b-a) = \int_a^b f(x)\,dx.$$

This means that the rectangle with base $[a, b]$ and height c has the same area as the region under the graph of f from a to b.

6 Example

The graph of some function f is given below. List the following values from *smallest* to *largest*.



1. $f'(1)$
2. The average value of $f(x)$ for $0 \leq x \leq a$.
3. The average value of the rate of change in $f(x)$ for $0 \leq x \leq a$.
4. $\int_0^a f(x)dx$.

References

- §6.3–6.4 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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