Math 1b. Lecture 13
Sequences

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1 Goals

- To understand and be able to the notion of a sequence of numbers.
- To understand and be able to apply the notions of iteratively defined and recursively defined sequences.
- To understand and be able to apply the definitions of convergence and divergence of a sequence.[1]
- To understand a be able to apply the various properties of sequences.
- To understand and be able to apply the fact that a bounded monotonic sequence converges.

2 What is a Sequence?

A sequence is a list of numbers, $a_1, a_2, a_3, \ldots$. More formally, we can think of a sequence as a function

$$a : \mathbb{N} \to \mathbb{R},$$

where $a(n) = a_n$. We often write $\{a_n\}_{n=1}^{\infty}$ for $a_1, a_2, a_3, \ldots$. We say

$$\lim_{n \to \infty} a_n = L$$

if we can make the terms a_n as close to L as we like by choosing n sufficiently large.

[1] We do not cover the formal definition of a convergent sequence.

Examples

- $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \to 0$
- $1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^{n-1}}, \ldots \to 0$
- $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, \frac{(-1)^{n+1}}{n}, \ldots \to 0$
- $1, 0, 1, \ldots, \frac{1+(-1)^{n+1}}{2}, \ldots$ diverges
- The Fibonnaci sequence, $1, 1, 2, 3, 5, 8, 13, \ldots$ or $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n = 3, 4, 5, \ldots$.
- $a_n = \sin n$.
- $a_0 = 0$ and $a_{n+1} = \sqrt{6 + a_n}$ for $n = 1, 2, 3, \ldots$.

3 Properties of Sequences

- If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, then
$$\lim_{n\to\infty} (a_n) = L.$$
- Algebraic properties; e.g.,
$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$
- If f is a continuous function and $\lim_{n\to\infty} a_n = L$, then
$$\lim_{n\to\infty} f(a_n) = f(L).$$
- The Squeeze Property.
- How to apply l'Hospital's Rule.

4 Monotonic and Bounded Sequences

Every bounded monotonic sequence is convergent. Consider the sequence $a_0 = 0$ and $a_{n+1} = \sqrt{6 + a_n}$ for $n = 1, 2, 3, \ldots$.

- The sequence is bounded by 3 (mathematical induction).

$$a_{n+1} = \sqrt{6 + a_n} \leq \sqrt{6 + 3} = 3.$$

- This sequence is monotonic since

$$\begin{aligned} 0 \geq (a_k - 3)(a_k + 2) &= a_k^2 - a_k - 6 \\ \Rightarrow 6 + a_k &\geq a_k^2 \\ \Rightarrow a_{n+1} = \sqrt{6 + a_n} &\geq a_k. \end{aligned}$$

Thus, $\lim_{n\to\infty} a_n = L$. Since

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6 + a_n} = \sqrt{\lim_{n\to\infty} 6 + a_n} = \sqrt{6 + L}$$

or $L = 3$.

5 Worksheet Problems

1. Give a example of a sequence that is:
   (a) convergent but not monotone.
   (b) monotone but not convergent.
   (c) bounded but not monotone.
   (d) monotone decreasing and unbounded.
   (e) monotone increasing and convergent.
   (f) unbounded but not monotone.
2. Find a_n for the sequence $-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots$.
3. Compute $\lim_{k\to\infty} a_k$ if it exists, where

$$a_k = \left(1 + \frac{2}{k}\right)^k$$

4. Compute $\lim_{k\to\infty} a_k$ if it exists, where

$$a_k = \ln k - \ln(3k+2).$$

5. Show that the sequence defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$

satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

References

- §8.2 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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