Math 1b. Lecture 15
Series—Definitions and Properties

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1 Goals

- To understand the definition of a series.
- To understand and be able to apply the basic properties of series.
- To understand and be able to apply the Divergence Test.

2 An Example

Suppose that the country of Pottsylvania spends \$2 billion and that each recipient of a fraction of this wealth spends 90% of the dollars that he or she receives. In turn, the secondary recipients spend 90% of the dollars that they receive, and so on. What is the total spending that results from the original injection of \$2 billion dollars into the economy?

$$2 + 0.9 \cdot 2 + (0.9)^2 \cdot 2 + (0.9)^3 \cdot 2 + \cdots$$

This is an example of an infinite series.

3 Definitions

- An infinite series is an infinite sum of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots + a_k + \cdots .$$

Consider the example

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \cdots .$$

If we draw a picture of a unit square and consider each term in the series to be an area, then it is easy to conclude that $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

- We need a formal definition of convergence. Let

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

be a *partial sum*. We say the series *converges* if

$$\lim_{n\to\infty} S_n = L$$

exists.

4 Examples

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ converges to 1. This is an example of a telescopic series.

- The *Harmonic series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Compare $1 + \displaystyle\int_1^{\infty} \frac{dx}{x}$ to a right-hand sum for the integral.

5 Properties

- $\displaystyle\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k$

- $\displaystyle\sum_{k=1}^{\infty} a_k \pm b_k = \sum_{k=1}^{\infty} a_k \pm \sum_{k=1}^{\infty} b_k$

6 Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

7 Worksheet Problems

1. Determine the convergence or divergence of each of the following series. If the series converges, try to find the sum.

   (a) $\displaystyle\sum_{n=1}^{\infty} \tan n$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

2. Let $\{a_n\}$ be an increasing sequence such that $a_1 > 0$ and $a_n \leq 100$ for all $n \geq 1$.

   (a) Does $\displaystyle\lim_{n\to\infty} a_n$ converge?

   (b) Does $\displaystyle\sum_{n=1}^{\infty} a_n$ converge?

3. Suppose that the partial sums of $\sum_{n=1}^{\infty} a_n$ satisfy the inequality

$$\frac{6\ln n}{\ln(n^2+1)} < S_n < 3 + ne^{-n}$$

   for all $n > 100$.

   (a) Does $\displaystyle\sum_{n=1}^{\infty} a_n$ converge? If so, what is the limit?

   (b) Can anything be said about $\displaystyle\lim_{n\to\infty} a_n$?

References

- §8.2 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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