Math 1b. Lecture 16
The Integral and Comparison Tests; Estimating Sums (The Integral Test and p-series)

T. Judson

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1 Goals

- To understand and be able to apply the integral test to find the convergence of a series.
- To understand and be able to tell if a p-series converges.

2 An Example

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges to $\pi^2/6$. While it requires some additional mathematics to determine this fact, we can, however, determine that the series converges to *something*,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\, dx.$$

3 The Integral Test

Suppose that f is continuous, positive, and decreasing on $[1, \infty)$ and $a_n = f(n)$.

1. If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

2. If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

To show this, note that

$$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx \le a_1 + a_2 + \cdots + a_{n-1}.$$

If $\int_1^\infty f(x)\,dx$ is convergent, then

$$\sum_{k=2}^{n} a_k \le \int_1^n f(x)\,dx \le \int_1^\infty f(x)\,dx$$

and

$$S_n = a_1 + \sum_{k=2}^{n} a_k \le a_1 \int_1^\infty f(x)\,dx = M.$$

Since $S_{n+1} = S_n + a_{n+1} \ge S_n$ and $S_n \le M$ for all n, the sequence of partial sums is bounded and increasing and must converge.

4 p-Series

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{x^p}$$

converges for $p > 1$ and diverges for $p \le 1$.

5 Worksheet Problems

1. Use the integral test to determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}.$$

2. Use the integral test to determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

3. Use the integral test to determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}.$$

4. If $f(k) = a_k$, rank the values $\int_1^n f(x)\,dx$, $\sum_{k=1}^{n-1} a_k$, and $\sum_{k=2}^{n} a_k$ in increasing order.

5. If $f(k) = a_k$, rank the values $\int_n^\infty f(x)\,dx$, $\sum_{k=n+1}^{\infty} a_k$, and $\int_{n+1}^\infty f(x)\,dx$ in increasing order.

6. For what values of p is the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

convergent?

References

- §8.3 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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