Math 1b. Lecture 17
The Integral and Comparison Tests; Estimating Sums (Comparison Tests)

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1 Goals[1]

- To understand and be able to apply the Comparison Test to determine the convergence or divergence of a series.
- To understand and be able to the remainder of a series using the Integral Test.

2 The Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ also converges.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ also diverges.

For example, suppose that we wish to determine the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 1 + \sin^2 n}.$$

Since

$$\frac{1}{n^2 + 2n + 1 + \sin^2 n} \leq \frac{1}{n^2}$$

for $n = 1, 2, \ldots$ and the series $\sum_{n=1}^{\infty} 1/n^2$ converges, the original series must also converge.

[1] We will not require that students know the Limit Comparison Test

3 Computing the Remainder of a Series

- Suppose that we wish to estimate $\sum_{n=1}^{\infty} 1/n^5$ to three decimal places.
- The remainder of a convergent series $S = \sum_{n=1}^{\infty} a_n$ with partial sums $S_N = \sum_{n=1}^{N} a_n$ is

$$R_N = S - S_n = a_{N+1} + a_{N+2} + \cdots .$$

  We know that

$$\int_{N+1}^{\infty} f(x)\,dx, \leq R_N \leq \int_{N}^{\infty} f(x)\,dx,$$

  where $f(k) = a_k$.

- Solving the inequality

$$R_N \leq \int_{N}^{\infty} \frac{1}{x^5}\,dx = \frac{1}{4N^4} \leq 0.0005,$$

  we can choose $N = 5$,

4 Worksheet Problems

1. Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2 - 1}{3n^4 + 1}.$$

2. Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2 - 1}{3n^4 + 1}.$$

3. Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{4 + 3^n}{2^n}.$$

4. Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1 + \sin n}{10^n}.$$

5. Estimate the series
$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$
to within 0.01.

6. If we estimate $\sum_{n=1}^{\infty} 1/n^{3/2}$ by
$$\sum_{n=1}^{10} \frac{1}{n^{3/2}},$$
how accurate is our estimate?

References

- §8.3 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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