Math 1b. Lecture 18
Other Convergence Tests (Alternating Series, Absolute and Conditional Convergence)

T. Judson

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1 Goals

- To understand and be able to apply the definition of an alternating series.
- To understand and be able to apply the concepts of conditional and absolute convergence.

2 Alternating Series

An alternating series is a series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots .$$

3 The Alternating Series Test

Let

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

be an alternating series satisfying the following conditions.

1. $a_1 \geq a_2 \geq a_3 \geq \cdots$
2. $\lim_{n \to \infty} a_n = 0$

Then the series converges.

For example, the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges.

4 Proof of the Alternating Series Test

Let

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots .$$

The sequence of even partial sums,

$$\begin{gathered} S_2 = a_1 - a_2 \geq 0 \\ S_4 = a_1 - a_2 + a_3 - a_4 = S_2 + a_3 - a_4 \geq S_2 \\ S_6 = S_4 + a_5 - a_6 \geq S_4 \\ \vdots \\ S_{2n} = S_{2n-2} + a_{2n-1} - a_{2n} \geq S_{2n-2}, \end{gathered}$$

is increasing. This sequence is also bounded since

$$S_{2n} = a_1 - (a_2 - a_3) - (a_2 - a_3) - \cdots - (a_{2n-1} - a_{2n-2}) - a_{2n} \leq a_1.$$

Thus, the sequence $\{S_{2n}\}$ is increasing and bounded above and

$$\lim_{n \to \infty} S_{2n} = S.$$

We claim that the limit of odd partial sums also converges to S,

$$\begin{aligned} \lim_{n \to \infty} S_{2n+1} &= \lim_{n \to \infty} S_{2n} + a_{2n+1} \\ &= \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} a_{2n+1} \\ &= S + 0 = S. \end{aligned}$$

5 Estimating Sums

If

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

is an alternating series satisfying

1. $a_1 \geq a_2 \geq a_3 \geq \cdots,$
2. $\lim_{n\to\infty} a_n = 0,$

then

$$|R_n| = |S - S_n| \leq a_{n+1}.$$

This is true since

$$|S - S_n| \leq |S_{n+1} - S_n| = a_{n+1}.$$

6 Absolute and Conditional Convergence

A series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$. If $\sum a_n$ converges, but $\sum |a_n|$ diverges, then $\sum a_n$ is *conditionally convergent*. For example,

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent.
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

An absolutely convergent series is convergent. Suppose that $\sum |a_n|$ converges. Then $\sum 2|a_n|$ is convergent. Since

$$0 \leq a_n + |a_n| \leq 2|a_n|,$$

the series $\sum(a_n + |a_n|)$ converges by the comparison test. Thus,

$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$$

converges.

7 Worksheet Problems

1. Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$.

2. Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 + 2^n}$.

3. Suppose that $a_n \geq 0$ for all $n \geq 1$. Is is possible for $\sum_{n=1}^{\infty} a_n$ to converge conditionally?

4. For what values of p does the series $\displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln k}{n^p}$ converge absolutely?

5. For what values of p does the series $\displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln k}{n^p}$ converge conditionally?

6. Determine the convergence of $1 - \frac{1}{2^3} + \frac{1}{3^2} - \frac{1}{4^3} + \frac{1}{5^2} - \frac{1}{6^3} + \frac{1}{7^2} - \frac{1}{8^3} + \cdots$.

References

- §8.4 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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