Math 1b. Lecture 21
Power Series

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1 Goals

- To understand and be able to apply the idea of a power series.
- To be able to determine the interval of convergence of a power series.

2 Definition of a Power Series

A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots .$$

A *power series centered at* $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots .$$

The most natural example of a power series is the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

3 Convergence of Power Series

We can usually find where a power series converges using the ratio test.

4 Theorem

For a given power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

there are three possibilities.

1. The series only converges for $x = a$.
2. The series converges for all x.
3. There is a positive number R called *the radius of convergence* such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

In case (3), the endpoints of the interval must be tested for convergence separately.

5 Examples

Series	Radius of Convergence R	Interval of Convergence
$\sum_{n=0}^{\infty} ax^n$	$R = 1$	$(-1, 1)$
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$R = \infty$	$(-\infty, \infty)$
$\sum_{n=0}^{\infty} n!x^n$	$R = 0$	$\{0\}$
$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
$\sum_{n=1}^{\infty} \frac{x^n}{n^2 5^n}$	$R = 5$	$[-5, 5]$

6 Worksheet Problems

1. Find the interval and radius of convergence of $\sum_{n=1}^{\infty}(x-2)^n$
2. Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges if $x = -3$ and diverges if $x = 7$. Indicate which of the following statements must be true, cannot be true, or may be true.

(a) The power series converges if $x = -10$.

(b) The power series diverges if $x = 3$.

(c) The power series converges if $x = 6$.

(d) The power series diverges if $x = 2$.

(e) The power series diverges if $x = -7$.

(f) The power series converges if $x = -4$.

3. Find the interval and radius of convergence of $\sum_{n=2}^{\infty} \frac{(x-3)^{2n}}{n^4}$

4. Find the interval and radius of convergence of $\sum_{n=2}^{\infty} \frac{(x+5)^n}{n \ln n}$

5. Find the interval and radius of convergence of $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n}$

6. Give an example of a power series that converges on the interval $[-11, -3)$.

References

- §8.5 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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