Math 1b. Lecture 22
Representations of Functions by Power Series

T. Judson

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1 Goals

- To understand and be able to represent functions as power series.
- To be able to differentiate and integrate power series to obtain new ways to represent functions as power series.

2 Representations of Functions by Power Series

- $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \ |x| < 1$
- $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots, \ |x| < 1$
- $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots, \ |x| < 1$
- $$\begin{aligned} \frac{x^2}{x+3} &= \frac{x^2}{3}\left(\frac{1}{1-(-x/3)}\right) \\ &= \frac{x^2}{3}\left(1 - \frac{x}{3} + \frac{x^2}{3^2} - \frac{x^3}{3^3} + \cdots\right) \\ &= \frac{x^2}{3} - \frac{x^3}{3^2} + \frac{x^4}{3^3} - \frac{x^5}{3^4} + \cdots \end{aligned}$$

3 Differentiating and Integrating Power Series

If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is term-by-term differentiable and integrable on the interval $(a-R, a+R)$ and

- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
- $\displaystyle\int f(x)\,dx = C + c_0x + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + c_3\frac{(x-a)^4}{4} + \cdots$
  $\displaystyle= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

4 Examples

- $\arctan x = \displaystyle\int \frac{1}{1+x^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$
- $\ln(x+1) = \int \frac{1}{1+x}\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

5 Worksheet Problems

Function	Series	Interval of Convergence
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$	$(-\infty, \infty)$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$	$(-\infty, \infty)$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$\frac{1}{1+x}$	$1 - x + x^2 - x^3 + \cdots$	$(-1, 1)$
$\frac{1}{1+x^2}$	$1 - x^2 + x^4 - x^6 + \cdots$	$(-1, 1)$
$\arctan x$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1, 1]$
$\ln(x+1)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1, 1)$

1. Find a power series representation for $f(x) = \frac{x^2}{1+x}$.
2. Find a power series representation for $f(x) = \frac{1}{(1+x)^2}$.
3. Find a power series representation for $f(x) = \frac{1}{2+x}$ and determine the interval of convergence.
4. Find a power series representation for $f(x) = x^2 \cos x$.
5. Find a power series representation for $f(x) = \cos(x^2)$.
6. Show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$.

References

- §8.6 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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