Math 1b. Lecture 23
Taylor and Maclaurin Series I

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1 Goals

- To understand and be able to compute Taylor and Maclaurin series.
- To understand and be able to find the Taylor polynomial of a function.

2 Does Every Function Have a Power Series Representation?

This is certainly not the case since a function must be infinitely differentiable. Thus, $f(x) = |x|$ has no power series representation at $x = 0$. If f does have a power series representation, then the following must be true.

If a function f has a power series representation at $x = a$,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R,$$

then the coefficients of the power series are given by

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

3 Taylor and Maclaurin Series

- A *Taylor Series* for f centered at $x = a$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots.$$

- A Taylor series centered at $x = 0$ is called a *Maclaurin series*,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Thus, for a given a function f that is infinitely differentiable, we can compute its Taylor series. The question of whether or not the series converges remains.

4 Taylor Polynomials

The nth degree *Taylor Polynomial* of a function f is

$$T_n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The Taylor polynomials approximate f near a. In other words, the Taylor polynomials are the partial sums of a Taylor series.

5 Worksheet Problems

1. Find the Taylor polynomial for $f(x) = \cos x$, where $n = 4$ and $a = 0$
2. Find the Taylor polynomial for $f(x) = \cos x$, where $n = 4$ and $a = \pi/6$
3. Find the Taylor series for $f(x) = \cos x$, where $a = 0$.
4. Find the Taylor series for $f(x) = x^3 - 3x^2 + 2$, where $a = 2$.
5. Find the Taylor series for $f(x) = \ln x$, where $a = 1$.

References

- §8.7 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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