Math 1b. Lecture 24
Taylor and Maclaurin Series II

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1 Goals

- To understand the convergence of a Taylor series.
- To understand and be able to compute Taylor's Inequality.

2 The Remainder of a Taylor Series

If f has a Taylor series and

$$T_n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the nth degree *Taylor Polynomial* of a function f, then

$$f(x) = \lim_{n\to\infty} T_n.$$

Define the *remainder* of a Taylor series to be

$$R_n(x) = f(x) - T_n(x).$$

If R is the radius of convergence of the series, then

$$\lim_{n\to\infty} R_n = 0$$

for $|x - a| < R$.

3 Taylor's Inequality

If $|f^{(n+1)}| \leq M$ between a and x, then the remainder of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

4 Taylor Series of Some Familiar Functions

- $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$
- $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$
- $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$

5 Multiplication and Division of Power Series

Power series can be multiplied and divided just like polynomials. For example,

$$e^x \sin x = x + x^2 + \frac{x^3}{3} + \cdots.$$

6 Worksheet Problems

1. Let
$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$
for all real numbers x.
   (a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local minimum, a local maximum, or neither at $x = 0$.
   (b) Show that $y = f(x)$ is a solution to the differential equation
   $$xy' + y = \cos x.$$
2. Let f be the function given by
$$f(x) = \sin\left(5x + \frac{\pi}{4}\right),$$

and let $P(x)$ be the third-degree Taylor polynomial for f centered about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the error bound to show that

$$\left| f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right| < \frac{1}{100}.$$

(d) Let

$$G(x) = \int_0^x f(t)\,dt.$$

Write the third-degree Taylor polynomial for G about $x = 0$.

3. If $f(x) = \cos x$ and $g(x) = \sin x$, find the first five terms of the power series for $f(x)g(x)$.

References

- §8.7 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

Notes

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