Math 1b. Lecture 26
Modeling with Differential Equations

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1 Goals

- To understand how differential equations can be used to model problems from the natural sciences, engineering, economics, and the social sciences.

2 Differential Equations

We can often describe interesting natural phenomena involving change with equations that relate the changing quantities. How a function changes is measured by the function's derivative, and an equation relating a function to one or more of its derivatives is called a *differential equation*. For example, one of the simplest differential equations is

$$\frac{dx}{dt} = kx.$$

It is not too difficult to see that $x(t) = Ce^{kt}$ is a *solution* to this equation, where C is an arbitrary constant. If we differentiate $x(t)$, we obtain

$$x'(t) = kCe^{kt} = kx(t).$$

If, in addition, we know the value of $x(t)$, say when $t = 0$, we can also determine the value of C. If $x(0) = x_0$, then

$$x_0 = x(0) = Ce^{k \cdot 0} = C$$

or $x(t) = x_0 e^{kt}$. The differential equation

$$\begin{aligned} x'(t) &= kx(t), \\ x(0) &= x_0 \end{aligned}$$

is an example of an *initial value problem.* We say that $x(0) = x_0$ is an *initial condition.* The expression, $x(t) = Ce^{kt}$ is a *general solution* of the equation $x' = kx$, and $x(t) = x_0e^{kt}$ is a *particular solution* to the differential equation. The general solution to our equation $x(t) = Ce^{kt}$ graphs as an infinite family of curves, which we will call *integral curves* (Fig 1).

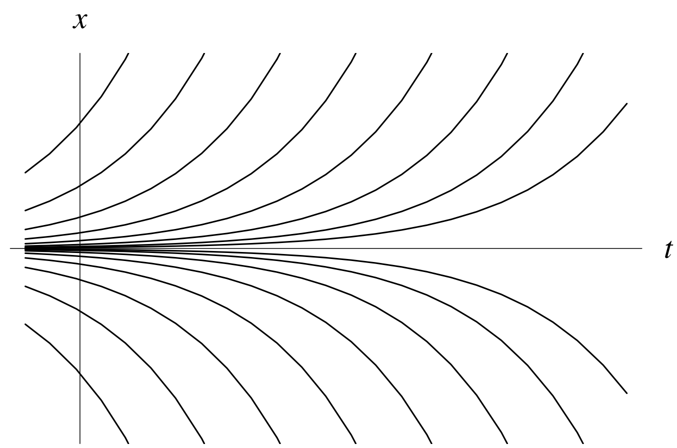

Figure 1: Integral Curves

3 Exponential Growth

Differential equations are very useful for modeling real world phenomena. For a particular situation that we might wish to investigate, our first task is to write an equation (or equations) that best describes the phenomenon. Suppose that we wish to study how a population, $P(t)$, grows at time t. We might make the assumption that a constant fraction of population is having offspring at any particular time. If we also assume that the population has a constant death rate, the change in the population during the interval Δt will be

$$\Delta P \approx k_{\text{birth}} P(t)\Delta t - k_{\text{death}} P(t)\Delta t,$$

where k_{birth} is the fraction of the population having children during the interval and k_{death} is the fraction of the population that dies during the interval. Therefore,

$$\frac{\Delta P}{\Delta t} \approx kP(t),$$

where $k = k_{\text{birth}} - k_{\text{death}}$. Since the derivative of P is

$$\frac{dP}{dt} = \lim_{\Delta \to \infty} \frac{\Delta P}{\Delta t},$$

the rate of change of the population is proportional to the size of the population,

$$\frac{dP}{dt} = kP$$

at time t. We already know that the general solution to this equation is $P(t) = Ce^{kt}$, and we say that the population grows *exponentially* in this case.

For example, suppose that $P(t)$ is a population of a colony of bacteria at time t, whose initial population is 1000 at $t = 0$, where time is measured in hours. Then

$$1000 = P(0) = Ce^0 = C,$$

and our solution becomes $P(t) = 1000e^{kt}$. If the population grows at three percent per hour, then

$$1030 = P(1) = 1000e^k,$$

after one hour. Consequently,

$$k = \ln 1.03 \approx 0.0296$$

and the solution to our initial value problem is

$$P(t) = 1000e^{0.0296t}.$$

4 Logistic Growth

Not all populations grow exponentially. For example, the population of fish in a lake might be limited by available resources and food supply. A small population of fish might begin to grow exponentially if the lake was very large and food was abundant, but the growth rate would decline as the availability of resources in the lake declined. We can use *logistic equation* to model population growth in a resource limited environment.[1]

To see how the logistic model works, we will try to adjust our model of exponential growth to account for the limited resources of the lake. We will make the following assumptions.

[1]The logistic model was first used by the Belgian biologist Verhulst in 1846 to predict the populations of Belgium and France.

- If the population of fish is small and there are abundant resources, the rate of growth will be approximately exponential,

$$\frac{dP}{dt} \approx kP.$$

- If N is the maximum population of fish that the lake can support, then any population larger than N will decrease. In other words,

$$\frac{dP}{dt} < 0$$

  for $P > N$. We say that N is the *carrying capacity* for the population.

Our assumptions suggest that we might try an equation of the form

$$\frac{dP}{dt} = kf(P)P,$$

where $f(P)$ is a function of P that is close to one if the population is small, but negative if the population is greater than N. The simplest function satisfying these properties is

$$f(P) = \left(1 - \frac{P}{N}\right).$$

Thus, the *logistic population model* is given by

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P.$$

Suppose that our lake will support 1000 fish, and the initial population of the lake is 100 fish. Then we must solve the initial value problem

$$\begin{aligned} \frac{dP}{dt} &= k\left(1 - \frac{P}{1000}\right)P \\ P(0) &= 100 \end{aligned}$$

in order to determine the number of fish in the lake at any time t. It is easy to verify that

$$P(t) = \frac{1000}{9e^{-1000kt} + 1}$$

is the solution to our initial value problem, and we will learn how to solve such equations in next section. If we know that the population of the lake is 200 fish after one year, we can determine that

$$k = -\frac{1}{1000}\ln\left(\frac{4}{9}\right) \approx 0.0008109,$$

and

$$P(t) = \frac{1000}{9e^{-0.8109t} + 1}$$

The graph of our solution certainly fits the situation that we are trying to model (Fig. 2).

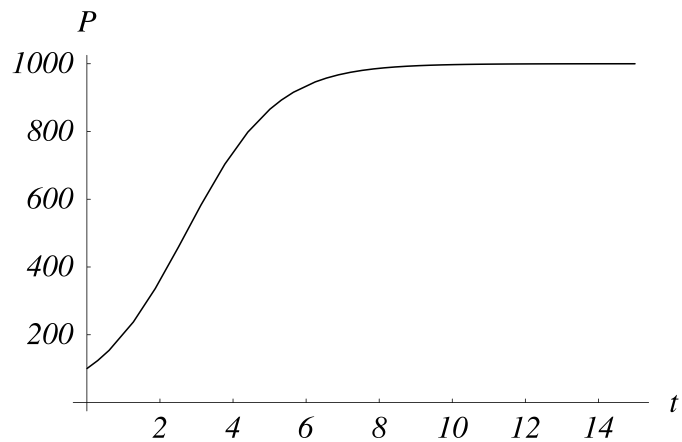

Figure 2: Logistic Growth

5 A Predator-Prey System

The Hudson Bay Company, founded in 1670 as a fur trading company, is Canada's oldest corporation. The company kept accurate records on the number of lynx (*lynx canadienis*) pelts that were bought from trappers from 1821 to 1940. The company noticed that the number of pelts varied from year to year and that the number of lynx pelts reached a peak about every ten years.[2]

The ten year cycle for lynx can be best understood using a model that incorporates the interaction between predators and their prey. This model was discovered independently by Lotka (1925) and Volterra (1926). The primary prey for the Canadian lynx is the snowshoe hare (*lepus americanis*). Let us denote the population of hares by $H(t)$ and the population of lynx by $L(t)$, where t is the time measured in years. We will make the following assumptions.

[2]Elton, C. S. and M. Nicholson. "The ten year cycle in the numbers of lynx in Canada," *Journal of Animal Ecology.* 1942 **11**(215–244).

- If no lynx are present, we will assume that the hares reproduce at a rate proportional to their population and are not affected by overcrowding. That is, the hare population will grow exponentially,

$$\frac{dH}{dt} = aH.$$

- One the other hand, the lynx prey on the hares. We can argue that the rate at which the hares are consumed by the lynx is proportional to the rate at which the hares and lynx interact. Thus, the equation that predicts the rate of change of the hare population becomes

$$\frac{dH}{dt} = aH - bHL.$$

  We thinking of HL as the number of possible interactions between the lynx and the hare populations.

- If there is no food, the lynx population will decline at a rate proportional to itself,

$$\frac{dL}{dt} = -cL.$$

- The lynx receive benefit from the hare population. The rate at which lynx are born is proportional to the number of hares that are eaten, and this is proportional to the rate at which the hares and lynx interact. Consequently, the growth rate of the lynx population can be described by

$$\frac{dL}{dt} = -cL + dHL.$$

We now have a *system* of differential equations that describe how the two populations interact,

$$\begin{aligned} \frac{dH}{dt} &= aH - bHL, \\ \frac{dL}{dt} &= -cL + dHL. \end{aligned}$$

Although we will learn how to analyze and solve systems of differential equations in later in the text; however, we will give a graphical solution in Figure 3 to the system

$$\begin{aligned} \frac{dH}{dt} &= 0.4H - 0.01HL, \\ \frac{dL}{dt} &= -0.3L + 0.005HL. \end{aligned}$$

Notice that the predator population begins to grow and reaches a peak after the prey population reaches its peak. As the prey population declines, the predator population also declines. Once the predator population is smaller, the prey population has a chance to recover that the cycle begins again.[3]

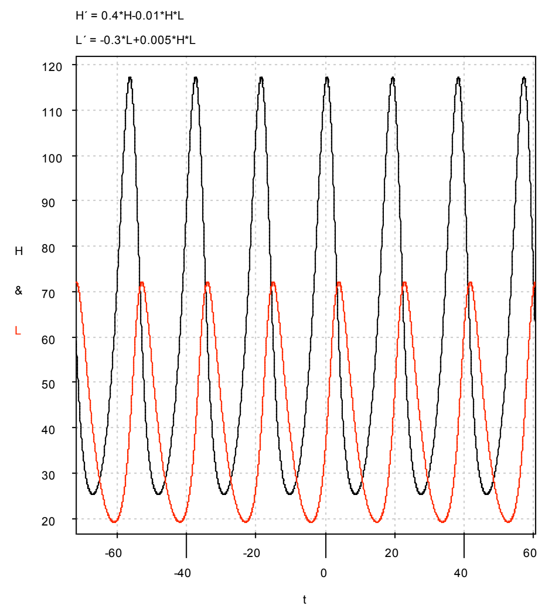

Figure 3: The predator-prey relationship between the lynx and the snowshoe hare

References

- §7.1 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

- §31.1 in Robin J. Gottlieb. *Calculus: An Integrated Approach to Functions and Their Rates of Change*, preliminary edition. Addison Wesley, Boston, 2002. ISBN 0–201–70929–5.

[3] An excellent account of the actual lynx and snowshoe hare data and model can be found in Brauer, F. and C. Castillo-Chávez. *Mathematical Models in Population Biology and Epidemiology*, Texts in Applied Mathematics 40. Springer, New York, 2001.

Notes

April 17, 2006