Math 1b. Lecture 27
Solutions to Differential Equations

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1 Goals

- To understand what it means to be the solution of a differential equation.
- To understand the idea of a slope field.
- To understand and be able to apply the existence and uniqueness theorem (weak form) for differential equations.
- To be able to solve differential equations of the form $dy/dt = g(t)$.
- To be able to solve differential equations of the form $dy/dt = ky$.
- To understand that autonomous equations of the form $dy/dt = f(y)$ are time independent.

2 Equations of the Form $dy/dt = g(t)$.

It is quite easy to solve an equation of the form $dy/dt = g(t)$—simply integrate both sides of the equation. However, equations of this form rarely occur in practice.

3 The Coffee Problem

Two identical cups of dark liquid are left in a 70°F laboratory cool. At time $t = 0$, the first cup's temperature was 190°F, and was dropping at a rate of 12°F per minute. When did this cup's temperature fall to 130°F? The second cup was at 130°F after 10 minutes. Could this liquid be coffee?

See if students can come up with the equation

$$\begin{aligned}\frac{dy}{dt} &= -\frac{1}{10}(y-70),\\ y(0) &= 190.\end{aligned}$$

4 Solutions of Differential Equations

A *solution* to a differential equation

$$\frac{dy}{dx} = f(x, y).$$

is a function $y = y(x)$ that satisfies the equation. For example, $y = x^4/4 + C$ is a solution to $y' = x^3$, where C is an arbitrary constant. If we specify an initial condition, $y(0) = y_0$, then we can find a unique solution.

In general, differential equations are very difficult to solve. However, it is quite easy to check if a function is a actually a solution. For example,

$$y(t) = 70 + 120e^{-0.1t}$$

is a solution to the differential equation

$$\begin{aligned}\frac{dy}{dt} &= -\frac{1}{10}(y-70),\\ y(0) &= 190.\end{aligned}$$

5 Newton's Law of Cooling

An object cools at a rate proportional to the temperature difference between the object and its environment. As a differential equation, Newton's Law of Cooling can be stated as

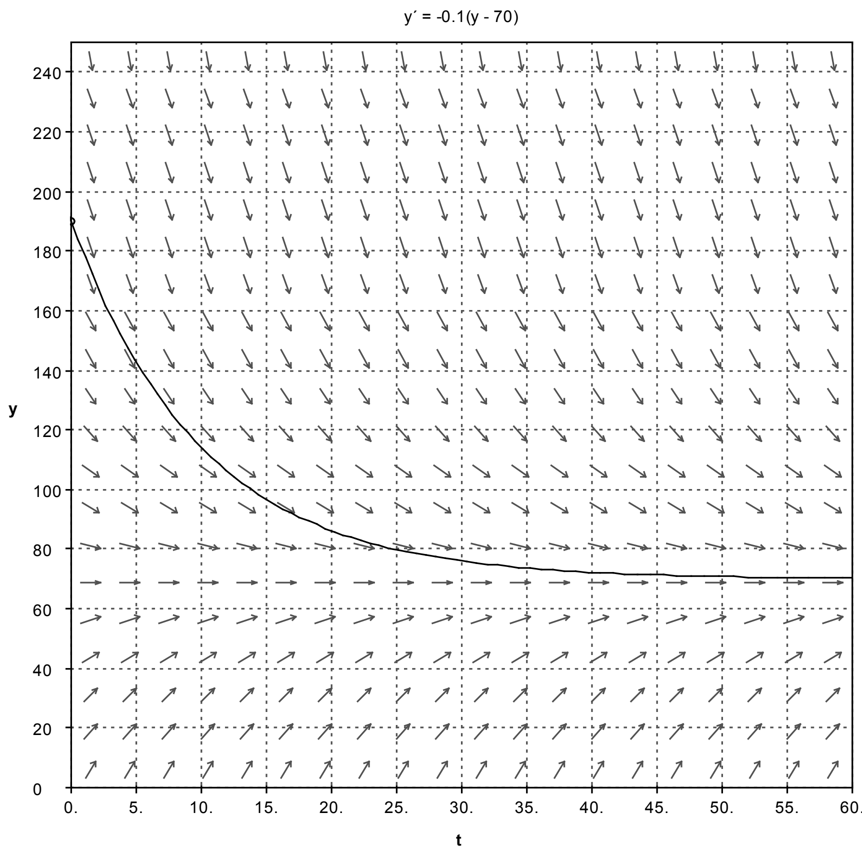
$$y' = k(y - T_e).$$

If we know the initial temperature at $t = 0$, then we have an *initial value problem*, which has a unique solution. We can state the initial condition as

$$y(0) = T_0.$$

6 Slope Fields

We can use slope fields to answer the coffee problem.



7 Existence and Uniqueness of Solutions

If an initial value problem has the form $y' = f(t)$ or $y' = g(y)$ and initial condition $y(a) = b$, then the initial value problem has a unique solution.

8 Worksheet Problems

1. Two identical cups of dark liquid are left in a 70°F laboratory cool. At time $t = 0$, the first cup's temperature was 190°F, and was dropping at a rate of 12°F per minute. When did this cup's temperature fall to 130°F? The second cup was at 130°F after 10 minutes. Could this liquid be coffee?

2. Decide whether the given function is a solution to the differential equation.

   (a) $y(t) = t^2/2; \qquad y' = t$

   (b) $y(t) = t^2/2; \qquad y' = y$

   (c) $y(t) = \frac{1}{2}t^4 + \frac{3}{2}t^2 + \frac{1}{4}; \qquad y' - \frac{2y}{t} = t^3$

   (d) $y(t) = \frac{1}{2}t^4 + \frac{3}{2}t^2; \qquad y' - \frac{2y}{t} = t^3$

3. Verify that

$$y(t) = 70 + 120e^{-0.1t}$$

   is a solution to the differential equation

$$\begin{aligned} \frac{dy}{dt} &= -\frac{1}{10}(y - 70), \\ y(0) &= 190. \end{aligned}$$

4. (8 points) Match equations and slope fields

   (a) $\frac{dy}{dt} = 1 + y^2$

   (b) $\frac{dy}{dt} = y^2 - t^2$

   (c) $\frac{dy}{dt} = ty$

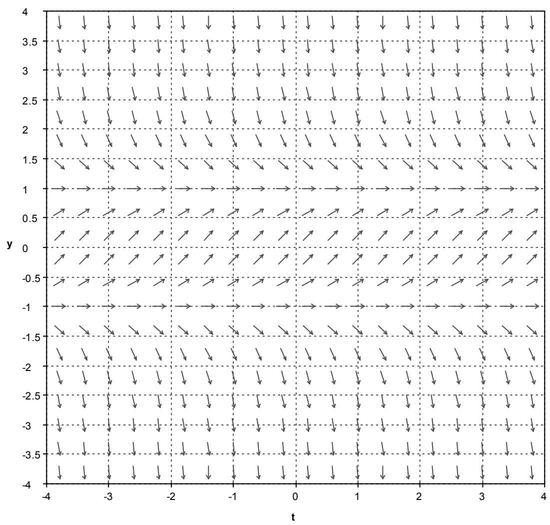
   (d) $\frac{dy}{dt} = 1 - y$

   (e) $\frac{dy}{dt} = y(1 - y) - 2$

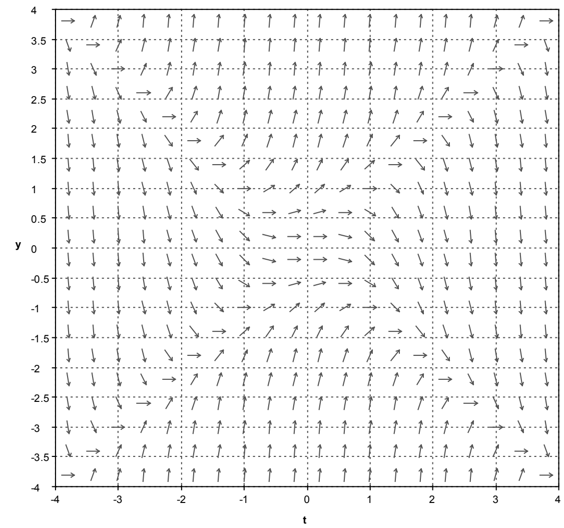
   (f) $\frac{dy}{dt} = (y - t)^2$

   (g) $\frac{dy}{dt} = 1 - y^2$

   (h) $\frac{dy}{dt} = y^2 - 4t^2$



(i)



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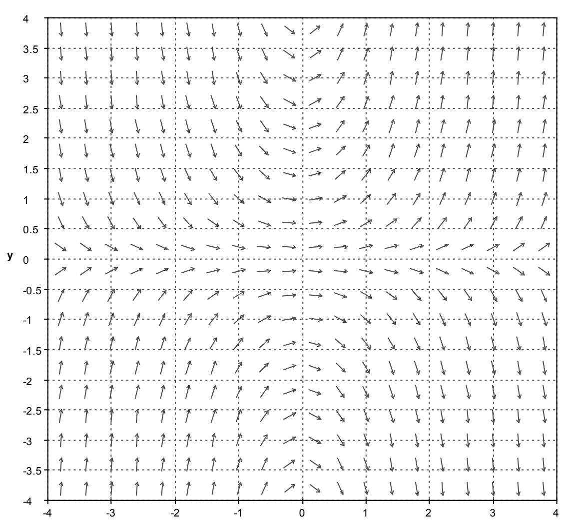

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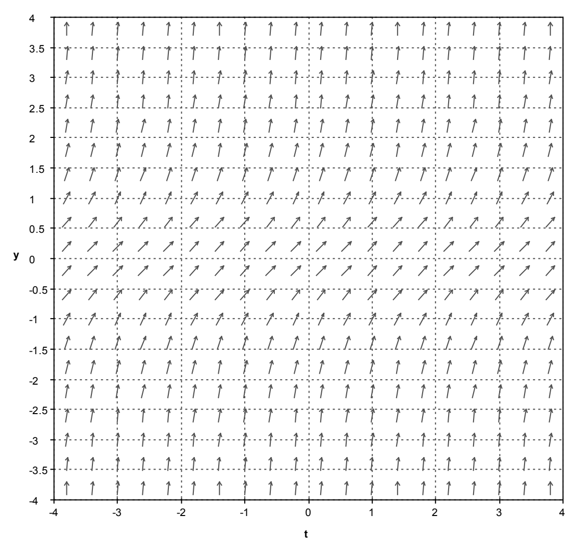

(iv)

References

- §7.1 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.

- §31.2 in Robin J. Gottlieb. *Calculus: An Integrated Approach to Functions and Their Rates of Change*, preliminary edition. Addison Wesley, Boston, 2002. ISBN 0–201–70929–5.

Notes

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