Math 1b. Lecture 31
Systems of Differential Equations (I)

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1 Goals

- To understand and be able to set up systems of first-order differential equations.
- To be understand and be able to analyze first-order systems by examining the phase plane.

2 Competing Species

Suppose that we have two species of fish in the same pond that compete for the same resources but do not prey upon one another. How can we model such a situation?

3 The Competing Species Model

Let x and y be the populations of two species at time t. We will assume that each species, in absence of the other, grows logistically:

$$\begin{aligned} x' &= x(a_1 - b_1 x) \\ y' &= y(a_2 - b_2 y), \end{aligned}$$

where a_1, a_2 are the growth rates of the two populations and a_1/b_1, a_2/b_2 are the carrying capacities.

If both species are present, each will impinge on the available food supply for the other. In effect, they reduce the growth rates and the saturation populations of the other. The simplest expression for reducing the growth rate of species x due to the presence of species y is to replace the growth factor $a_1 - b_1 x$ with $a_1 - b_1 x - \alpha_1$, where is a measure of the degree to which species y interferes with species x. The new system is now

$$\begin{aligned} x' &= x(a_1 - b_1 x) - \alpha_1 xy \\ y' &= y(a_2 - b_2 y) - \alpha_2 xy. \end{aligned}$$

The constants a_i, b_i, and α_i depend on the species of course.

4 Left and Right-Curling Snails

Discuss the near absence of left-curling snails discussed by Gould. Is the competing species a good model of this situation?

Let $L(t)$ denote the number (in millions) of left curling snails at time t and $R(t)$ the number of right-curling snails. The two populations compete for the same resources and might be governed by the following system of differential equations.[1]

$$\begin{aligned} \frac{dR}{dt} &= R - (R^2 + aRL) \\ \frac{dL}{dt} &= L - (L^2 + aLR). \end{aligned}$$

What information can be obtained from a differential equation without having to solve the equation? We have two very different cases depending on the parameter a.

The Case $a > 1$

Suppose $a = 2$. Then

$$\begin{aligned} \frac{dR}{dt} &= R - (R^2 + 2RL) \\ \frac{dL}{dt} &= L - (L^2 + 2LR). \end{aligned}$$

We are interested in what happens as $t \to \infty$. We will look at what happens when $dR/dt = 0$ and $dL/dt = 0$:

$$0 = \frac{dR}{dt} = R(1 - R - 2L)$$
$$0 = \frac{dL}{dt} = L(1 - L - 2R).$$

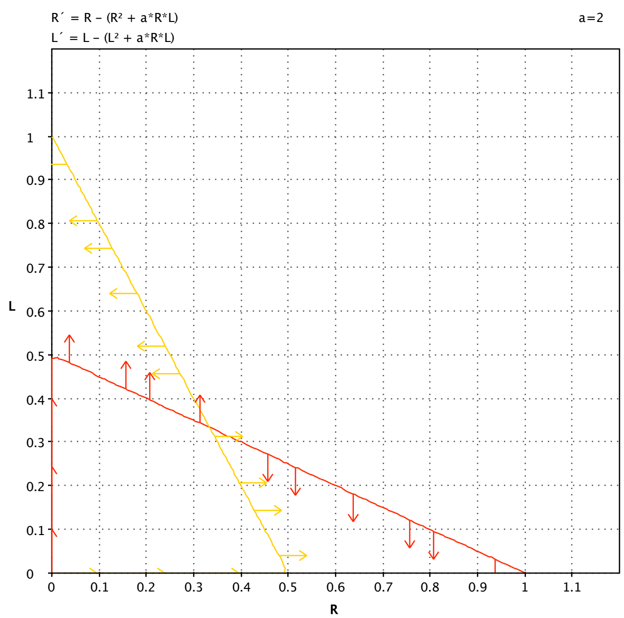
[1] We let $a_i = b_i = 1$ and $\alpha_1 = \alpha_2 = a$ in our model.

The R *null clines* are

$$\begin{aligned} R &= 0 \\ L &= (1-R)/2. \end{aligned}$$

The L *null clines* are

$$\begin{aligned} L &= 0 \\ L &= 1-2R. \end{aligned}$$



The null clines $dR/dt = 0$ and $dL/dt = 0$ intersect at $(0,0)$, $(1,0)$, $(0,1)$, and $(1/3, 1/3)$. These are called the *equilibrium points* of the system. We can choose a representative point in each region to find how the direction field is oriented.

What happens to the initial condition $(0.51, 0.50)$? We see three possible scenarios.

- Only left-curling snails survive and right-curling snails are extinct.
- Only right-curling snails survive and left-curling snails are extinct.
- There are essentially equal numbers of right and left curling snails.

The Case $a < 1$

Suppose $a = 1/2$. Then

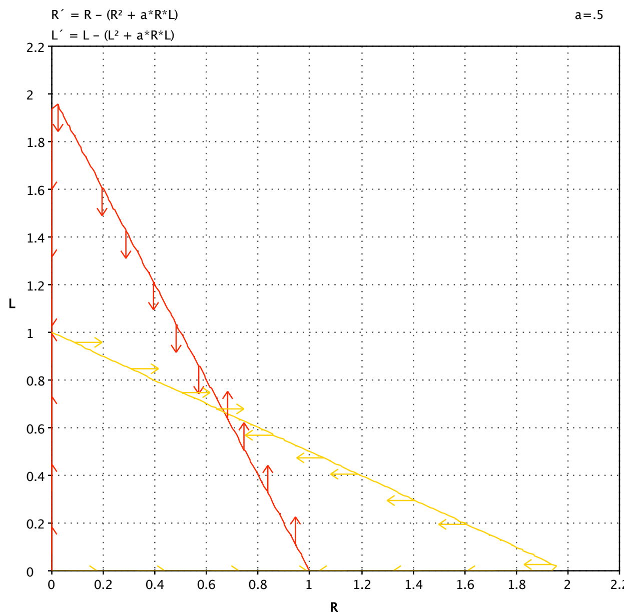
$$\begin{aligned} \frac{dR}{dt} &= R - (R^2 + RL/2) \\ \frac{dL}{dt} &= L - (L^2 + LR/2). \end{aligned}$$

The R *null clines* are

$$\begin{aligned} R &= 0 \\ L &= 2(1 - R). \end{aligned}$$

The L *null clines* are

$$\begin{aligned} L &= 0 \\ L &= 1 - R/2. \end{aligned}$$



The equilibrium points are $(0,0)$, $(1,0)$, $(0,1)$, and $(2/3,2/3)$. These are called the *equilibrium points* of the system. We can choose a representative point in each region to find how the direction field is oriented.

What happens to the initial condition $(0.51,0.50)$? No matter where we begin, we end up at $(2/3,2/3)$; hence, this is not a good model for right and left-curling snails.

5 The Lotka-Volterra Equation

Suppose we have a population of rabbits, R, and foxes, F. The system

$$\begin{aligned} \frac{dR}{dt} &= (a - bR - cF)R \\ \frac{dF}{dt} &= (-d + eR)F. \end{aligned}$$

models the predator-prey relationship between the foxes and rabbits. The first equation is of the form

$$\frac{dR}{dt} = \alpha R,$$

where $\alpha = a - bR - cF$. If $F = 0$ and there are no foxes, then this reduces to the logistic model. So a is the intrinsic growth rate of the rabbits in an unrestricted environment and a/b is the carrying capacity. The term $-cFR$ is the predatory effect of the foxes on the rabbit population and has a greater effect if the number of foxes increase.

The second equation is of the form

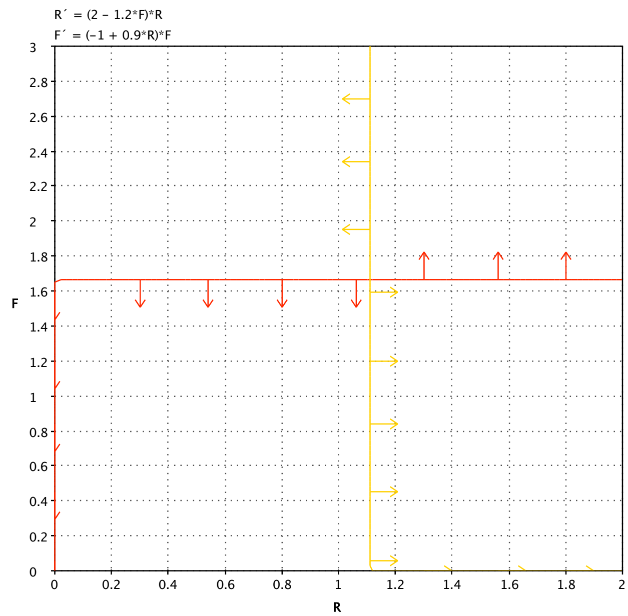
$$\frac{dF}{dt} = \beta F,$$

where $\beta = -d + eR$. If there are no rabbits, then the fox population is in decline and will eventually die out. The eFR term measures the positive effect that the rabbit population has on the birth rate of the foxes.

Consider the following systems.

- $$\begin{aligned} \frac{dR}{dt} &= (2 - 1.2F)R \\ \frac{dF}{dt} &= (-1 + 0.9R)F. \end{aligned}$$

  The R null clines for this system are $R = 0$ and $F = 1.67$. The F null clines are $F = 0$ and $R = 1.11$. We have closed or nearly closed orbits for solutions.



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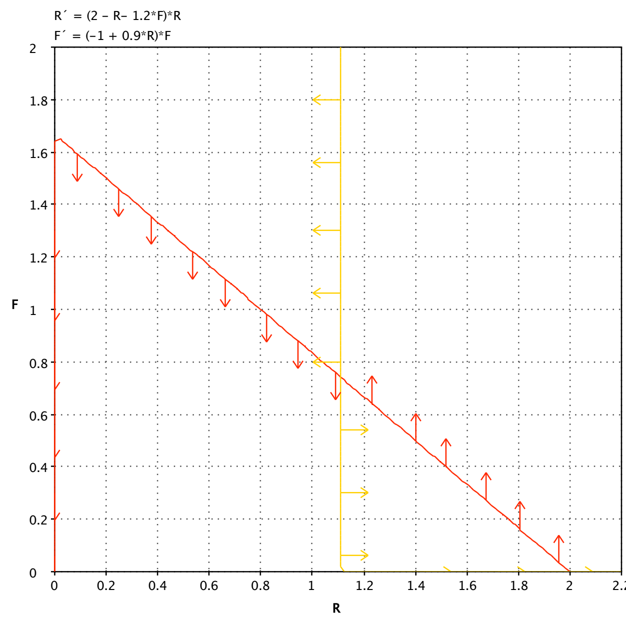
$$\frac{dR}{dt} = (2 - R - 1.2F)R$$
$$\frac{dF}{dt} = (-1 + 0.9R)F.$$

The R null clines for this system are

$$F = \frac{2 - R}{1.2}$$
$$R = 0.$$

The F null clines are $F = 0$ and $R = 1.11$. Solutions tend toward the intersection of the null clines.



References

- §7.6 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.
- §31.5 in Robin J. Gottlieb. *Calculus: An Integrated Approach to Functions and Their Rates of Change*, preliminary edition. Addison Wesley,

Boston, 2002. ISBN 0–201–70929–5.

Notes

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