# Math 1b. Lecture 32
# Systems of Differential Equations (II)

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## 1 Goals

- To understand and be able to model an epidemic in a closed population.
- To understand and to be able to apply phase plane analysis to

$$\begin{aligned}\frac{dx}{dt} &= f(x, y),\\ \frac{dy}{dt} &= g(x, y).\end{aligned}$$

- To understand that the system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y),\\ \frac{dy}{dt} &= g(x, y),\end{aligned}$$

  is completely predictive. If you choose a starting point in the $xy$-plane, then there is exactly one solution that starts at your chosen point.

- To be able to use *pplane*.

## 2 An Epidemic Model

Consider the model of a viral epidemic that moves through an isolated population. We make the following assumptions.

(a) Individuals are infected at a rate proportional to the product of the number of infected and susceptible individuals. We assume that the constant of proportionality is $\lambda = 0.05$ per day.

(b) The length of the incubation period is negligible. Infectious individuals are immediately infectious.

(c) On the average, an infected individual dies or recovers after 10 days.

(d) No one is sick initially.

(e) Infected individuals do not give birth, but susceptible individuals have a birth rate of 0.0003 per individual per year. Newborns are susceptible.

If $x(t)$ is the number of susceptible and $y(t)$ is the number of infected people, then

$$\begin{aligned}\frac{dx}{dt} &= -\lambda xy + 0.0003x \\ \frac{dy}{dt} &= \lambda xy - 0.1y.\end{aligned}$$

The first equation asserts that in a unit time interval, any infected individual will infect any given susceptible individual with $\lambda = 0.05$ percent probability. The constant $\lambda$ is a measure of the relative infectivity of the disease. If $\lambda$ is high relative to the birth rate then the disease will burn itself out.

### The Case $\lambda = 0.05$

The $x$ null clines are at $x = 0$ and $y = 0.006$. The $y$ null clines are at $y = 0$ and $x = 2$. The disease burns itself out.

### The Case $\lambda = 10^{-6}$

The $x$ null clines are at $x = 0$ and $y = 300$. The $y$ null clines are at $y = 0$ and $x = 100,000$. Note that the infection rate is much smaller compared to the birth rate. If we take our initial value of $x = 20,000$ and $y = 100$ $x$ will increase and $y$ will decrease (but will never become zero). Eventually, the number of infected individuals will hit 100,000, and the epidemic will take off. We will either get a situation where the disease is cyclic or it will spiral into an equilibrium point.

## 3 Phase Plane Analysis

- *Step 1.* Draw the curves where $f(x, y) = 0$. These curves are called the $x$ *null clines*. When $\mathbf{v}(t)$ lies on one of these curves, $dx/dt = 0$. Draw vertical slash marks on the $x$ null clines to remind yourself that a trajectory crossing the null cline can only do so if it is moving in a vertical direction at the instant of crossing.

- *Step 2.* Draw the curves where $y(x, y) = 0$. These curves are called the $y$ *null clines*. When $\mathbf{v}(t)$ lies on one of these curves, $dy/dt = 0$. Draw horizontal slash marks on the $y$ null clines to remind yourself that a trajectory crossing the null cline can only do so if it is moving in a horizontal direction at the instant of crossing.

- *Step 3.* Label the points where the $x$ and $y$ null clines intersect. These intersections are called *equilibrium points*. If $\mathbf{v}(t)$ is ever at one of these points , then both $dx/dt$ and $dy/dt$ vanish. This means that the trajectory stays at the point for all time. If the system is going to settle into a steady state, then $\mathbf{v}(t)$ will approach on of the equilibrium points as $t \to \infty$.

- *Step 4.* Label the regions of the $xy$-plane where $dx/dt < 0$ and where $dx/dy > 0$. These regions are always separated by $x$ null clines. Likewise, label the regions where $dy/dt$ is positive and negative.

- *Step 5.* Go back and put arrows on the vertical hash marks of the $x$ null clines. These arrows indicate whether the motion across the null cline is up or down. The arrows are up on the parts of the $x$ null cline that are in the $dy/dt > 0$ region, and down on those parts of the $x$ null cline in the $dy/dt < 0$ regions. Likewise, draw arrows on the horizontal slash marks of the $y$ null clines. These arrows are pointing right on the parts of the $y$ null cline in the $dx/dt > 0$ regions and left point on the parts in the $dx/dt < 0$ regions.

- *Step 6.*

  (a) If $dx/dt > 0$ and $dy/dt > 0$, then both $x(t)$ and $y(t)$ are increasing and the trajectory moves up and right.

  (b) If $dx/dt > 0$ and $dy/dt < 0$, the trajectory moves down and right.

  (c) If $dx/dt < 0$ and $dy/dt > 0$, the trajectory moves up and left.

  (d) If $dx/dt < 0$ and $dy/dt < 0$, the trajectory moves down and left.

## 4 Existence and Uniqueness

The system of equations

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned} \tag{1}$$

is completely predictive. If you choose a starting point in the $xy$-plane, then there is exactly one solution to the system that starts at your chosen point.

- For any starting point in the $xy$-plane, there is a unique solution to (1).
- Think of $\mathbf{v}(t)$ as tracing out a trajectory in the $xy$-plane as $t$ increases. The goal is to predict the behavior of this trajectory.
- The phase plane analysis is done to help predict the trajectory.

## References


- §7.6 in James Stewart. *Single Variable Calculus: Concepts & Context*, third edition. Brooks/Cole, Belmont CA, 2005. ISBN 0–534–41022–7.
- §31.5 in Robin J. Gottlieb. *Calculus: An Integrated Approach to Functions and Their Rates of Change*, preliminary edition. Addison Wesley, Boston, 2002. ISBN 0–201–70929–5.


## Notes

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