

Math 1b. Lecture 33  
Second-Order Homogeneous Differential Equations  
with Constant Coefficients (I)

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## 1 Goals

- To understand how use the equation  $mx'' + bx' + kx = 0$  to model a harmonic oscillator.
- To be able to solve equations of the form  $ax'' + bx' + cx = 0$ , when the roots of the characteristic equations are real and distinct.

## 2 A Spring-Mass Model

Sometimes it is necessary to consider the second derivative when constructing a mathematical model. Suppose that we have a mass lying on a flat surface and that this mass is attached to one end of a spring with the other end of the spring attached to a wall. We will denote the spring displacement by  $x$ . If  $x > 0$ , then the spring is stretched. If  $x < 0$ , the spring is compressed. If  $x = 0$ , then the spring is in a state of equilibrium (Fig. 1). If the surface is frictionless and we pull on the mass, then the mass will oscillate. We can construct a differential equation that will model our oscillating mass.

First, we must consider the restorative force on the spring. We will make the assumption that this force depends on the displacement of the spring,  $F(x)$ . Using Taylor's Theorem to expand  $F$ , we have

$$\begin{aligned} F(x) &= F(0) + F'(0)x + \frac{1}{2}F''(0)x^2 + \cdots \\ &= -kx + \frac{1}{2}F''(0)x^2 + \cdots, \end{aligned}$$

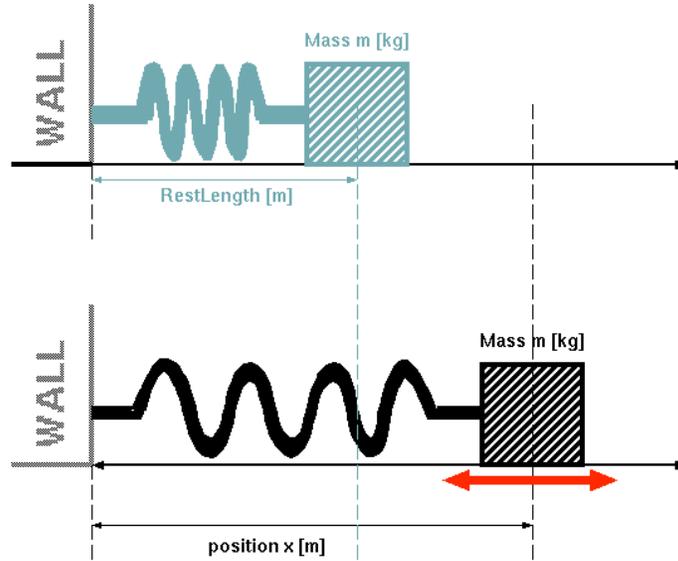


Figure 1: A Spring-Mass System

where  $F'(0) = -k$  and  $F(0) = 0$ . If the displacement is not too large, we can ignore higher ordered terms and consider the restorative force on the spring to be proportional to displacement of the spring from its equilibrium length,

$$F = -kx.$$

This is called *Hooke's Law*. We can test this law experimentally, and it is reasonably accurate if the displacement of the spring,  $x$ , is not too large.

By Newton's second law of motion, the force on the mass must be

$$F = ma = m \frac{d^2x}{dt^2} = mx''.$$

Setting the two forces equal, we have a *second order* differential equation,

$$mx'' = -kx$$

which describes our oscillating mass.

For example, suppose that we have a spring-mass system where  $m = 1$  and  $k = 1$ . If the initial velocity of the spring is one unit per second and

the initial position is at the equilibrium point, then we have the following initial value problem

$$\begin{aligned}x'' + x &= 0 \\x(0) &= 1 \\x'(0) &= 0.\end{aligned}$$

We might guess that a general solution of our differential equation could be of the form

$$x(t) = A \cos t + B \sin t,$$

since  $x''(t) = -x(t)$  for both sine and cosine. Noting that

$$x'(t) = -A \sin t + B \cos t,$$

and using our initial conditions, we can immediately determine that  $A = 1$  and  $B = 0$  or

$$x(t) = \cos t.$$

If there is friction in our system, it would act as a damping force. For example, our spring-mass system might be immersed in a viscous liquid. The simplest assumption would be to take the damping force to be proportional to the velocity of the mass,  $x'(t)$ . Thus, we would have an additional force,

$$F = -bx'$$

acting on our mass, where  $b > 0$ . Our new equation for the spring-mass system is

$$mx'' = -bx' - kx.$$

or

$$mx'' + bx' + kx = 0,$$

where  $m$ ,  $b$ , and  $k$  are all positive constants.

### 3 Guessing a Solution

Solutions of a linear system  $\mathbf{x}' = A\mathbf{x}$  often include terms of the form  $e^{rt}$ , so if we consider the equation

$$x'' + 7x' + 10x = 0, \tag{1}$$

it makes sense to guess that we can find a solution of the form  $e^{rt}$  as long as we choose  $r$  correctly. Substituting our guess of  $e^{rt}$  into the left-hand side of (1), we have

$$\begin{aligned}\frac{d^2}{dt^2}e^{rt} + \frac{d}{dt}7e^{rt} + 10e^{rt} &= r^2e^{rt} + 7re^{rt} + 10e^{rt} \\ &= (r^2 + 7r + 10)e^{rt} \\ &= (r + 5)(r + 2)e^{rt}.\end{aligned}$$

Since  $e^{rt}$  is never zero, we find that  $(r + 5)(r + 2) = 0$  or  $r = -5$  or  $-2$ . Thus, we have two solutions

$$x_1(t) = e^{-5t} \text{ and } x_2(t) = e^{-2t}.$$

By the Principle of Superposition,

$$x(t) = c_1x_1(t) + c_2x_2(t) = c_1e^{-5t} + c_2e^{-2t}$$

is a solution to  $x'' + 7x' + 10x = 0$ . Indeed, this is the general solution of our second-order equation since we have a one-to-one correspondence between the solutions of

$$x'' + 7x' + 10x = 0$$

and the system

$$\begin{aligned}x' &= y \\ y' &= -10x - 7y.\end{aligned}$$

Since  $x' = y$ , the solution to our system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1e^{-5t} \begin{pmatrix} 1 \\ -5 \end{pmatrix} + c_2e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

In general, suppose that

$$ax'' + bx' + cx = 0,$$

where  $a \neq 0$  and  $b^2 - 4ac > 0$ . If the roots of  $ar^2 + br + c$  are  $r_1$  and  $r_2$ , the general solution to the differential equation is

$$x(t) = c_1e^{r_1t} + c_2e^{r_2t}.$$

## 4 Worksheet Problems

1. Find the general solution of the equation  $y'' + 5y' + 6y = 0$
2. Find the general solution of the equation  $2y'' - 3y' - 2y = 0$
3. Find the general solution of the equation  $6y'' + 5y' - 6y = 0$
4. Solve the initial value problem

$$\begin{aligned}y'' - y' - 2y &= 0 \\y(0) &= -1 \\y'(0) &= 2\end{aligned}$$

5. Solve the initial value problem

$$\begin{aligned}10y'' - y' - 3y &= 0 \\y(0) &= 1 \\y'(0) &= 0\end{aligned}$$

6. Solve the initial value problem

$$\begin{aligned}y'' - 2y' - 3y &= 0 \\y(0) &= 2 \\y'(0) &= -3\end{aligned}$$

## References

- §31.6 in Robin J. Gottlieb. *Calculus: An Integrated Approach to Functions and Their Rates of Change*, preliminary edition. Addison Wesley, Boston, 2002. ISBN 0-201-70929-5.

## Notes

May 3, 2006