Math 1b. Lecture 34
Second-Order Homogeneous Differential Equations with Constant Coefficients (II)

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1 Goals

- To be able to solve equations of the form $ax'' + bx' + cx = 0$, when the roots of the characteristic equations and repeated and real or complex.

2 The Complex Case

Now let us solve the initial value problem

$$\begin{aligned} x'' + 4x' + 5x &= 0 \\ x(0) &= 3 \\ x'(0) &= -2. \end{aligned}$$

Again, we will assume that our solution has the form $x(t) = e^{rt}$. Substituting this function into our differential equation, we find that

$$0 = x'' + 4x' + 5x = r^2 e^{rt} + 4re^{rt} + 5e^{rt} = (r^2 + 4r + 5)e^{rt}.$$

As with the real case, $r^2 + 4r + 5 = 0$ or

$$r = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i.$$

Thus, we can find a solution

$$x(t) = e^{(-2+i)t} = e^{-2t} e^{it} = e^{-2t}(\cos t + i \sin t).$$

The real and imaginary parts of this solution are also solutions to our differential equation. Therefore, the general solution to our equation is

$$x(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t.$$

Applying our initial conditions, we can quickly calculate $c_1 = 3$ and $c_2 = 4$. Hence, the solution to our initial value problem is

$$x(t) = 3e^{-2t} \cos t + 4e^{-2t} \sin t.$$

If $b^2 - 4ac < 0$, the differential equation

$$ax'' + bx' + cx = 0$$

has a general solution

$$x(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t,$$

where $\alpha \pm i\beta$ are the roots of $ar^2 + br + c = 0$.

3 Repeated Roots

Finally, we will consider the equation

$$x'' + 2x' + x = 0.$$

If we choose e^{rt} as our guess, we find

$$\begin{aligned} x'' + 2x' + x &= r^2 e^{rt} + 2re^{rt} + e^{rt} \\ &= e^{rt}(r+1)^2 \\ &= 0. \end{aligned}$$

Thus, $r = -1$ and we have a solution $x_1(t) = e^{-t}$.

In order to find a general solution to $x'' + 2x' + x = 0$, we must find a second solution that is not a multiple of $x_1(t) = e^{-t}$. Since we already know that $cx_1(t)$ is a solution to our differential equation, we will try to generalize this observation by replacing c with a function $v(t)$ and then try to determine $v(t)$ so that $v(t)x_1(t)$ is a solution to $x'' + 2x' + x = 0$. Indeed, if

$$x(t) = v(t)x_1(t) = v(t)e^{-t},$$

then

$$x'(t) = v(t)x_1'(t) + v'(t)x_1(t) = -v(t)e^{-t} + v'(t)e^{-t}$$

and

$$\begin{aligned} x''(t) &= v''(t)x_1(t) + 2v'(t)x_1'(t) + v(t)x_1''(t) \\ &= v''(t)e^{-t} - 2v'(t)e^{-t} + v(t)e^{-t}. \end{aligned}$$

Consequently,

$$\begin{aligned} x'' + 2x' + x &= [v''e^{-t} - 2v'e^{-t} + ve^{-t}] + 2[-ve^{-t} + v'e^{-t}] + [ve^{-t}] \\ &= e^{-t}v'' \\ &= 0, \end{aligned}$$

and $v'' = 0$. Therefore, $v = c_1 t + c_2$. Letting $c_1 = 1$ and $c_2 = 0$, we can assume that $v(t) = t$, and the second solution to our equation is $x = te^{-t}$. Hence, the general solution to $x'' + 2x' + x = 0$ is

$$x(t) = c_1 e^{-t} + c_2 te^{-t},$$

which agrees with the solution that we would obtain from solving the system equivalent to this second-order differential equation.

4 Worksheet Problems

1. Find the general solution of the equation $y'' + 4y = 0$
2. Find the general solution of the equation $y'' + 2y' + 17y = 0$
3. Find the general solution of the equation $y'' - 6y' + 9y = 0$
4. Solve the initial value problem

$$\begin{aligned} y'' + 2y' + 3y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

5. Solve the initial value problem

$$\begin{aligned} y'' - 4y' + 13y &= 0 \\ y(0) &= 4 \\ y'(0) &= 0 \end{aligned}$$

6. Solve the initial value problem

$$\begin{aligned} y'' + 10y' + 25y &= 0 \\ y(0) &= 2 \\ y'(0) &= -1 \end{aligned}$$

References

- §31.6 in Robin J. Gottlieb. *Calculus: An Integrated Approach to Functions and Their Rates of Change*, preliminary edition. Addison Wesley, Boston, 2002. ISBN 0–201–70929–5.

Notes

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