Math 1b. Applications of Taylor Polynomials

Spring 2006

Taylor's Remainder Theorem and Taylor's Inequality[1]

- If $|f^{(n+1)}| \leq M$ between a and x, then the remainder of the Taylor series satisfies the inequality
$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$
- If f has continuous derivatives of orders $1, 2, \ldots, n+1$ between a and x, then
$$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$
where c lies between a and x.

  Notice that the Mean Value Theorem is a special case of Taylor's Remainder Theorem.

1. Compute the Taylor polynomial of $f(x) = \sqrt{x}$ for $n = 2$ and $a = 4$. Estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the interval $4 \leq x \leq 4.2$.

[1] This material is different than what is presented in Stewart.

2. Compute the Taylor polynomial of $f(x) = \cos x$ for $n = 4$ and $a = \pi/3$. Estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the interval $0 \le x \le 2\pi/3$.

3. Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by

$$T_3(x) = 7 - 9(x-2)^2 - 3(x-2)^3.$$

   (a) Find $f(2)$ and $f'(2)$.

   (b) Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative minimum, a relative maximum, or neither. Justify your answer.

   (c) Use $T_3(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative minimum, a relative maximum, or neither. Justify your answer.

   (d) The fourth derivative of f satisfies the inequality

$$\left|f^{(4)}(x)\right| \le 6$$

   for all x in the closed interval $[0, 2]$. Find an error bound on the approximation for $f(0)$ that you found in part (c). Explain why $f(0)$ must be negative.