20. $\dfrac{x^2+2x-1}{x^3-x} = \dfrac{x^2+2x-1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get $x^2+2x-1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$. Substituting 0 for x gives $-1 = -A \iff A = 1$. Substituting -1 for x gives $-2 = 2B \iff B = -1$. Substituting 1 for x gives $2 = 2C \iff C = 1$. Thus,
$$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int\left(\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1}\right)dx = \ln|x| - \ln|x+1| + \ln|x-1| + C = \ln\left|\frac{x(x-1)}{x+1}\right| + C.$$

23. $\dfrac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get
$x^3+x^2+2x+1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1) \iff$
$x^3+x^2+2x+1 = (Ax^3+Bx^2+2Ax+2B) + (Cx^3+Dx^2+Cx+D) \iff$
$x^3+x^2+2x+1 = (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$. Comparing coefficients gives us the following system of equations:

$$A + C = 1 \quad \textbf{(1)} \qquad B + D = 1 \quad \textbf{(2)}$$
$$2A + C = 2 \quad \textbf{(3)} \qquad 2B + D = 1 \quad \textbf{(4)}$$

Subtracting equation **(1)** from equation **(3)** gives us $A = 1$, so $C = 0$. Subtracting equation **(2)** from equation **(4)** gives us $B = 0$, so $D = 1$. Thus, $I = \displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx = \int\left(\frac{x}{x^2+1} + \frac{1}{x^2+2}\right)dx$. For $\displaystyle\int \frac{x}{x^2+1}\,dx$, let $u = x^2+1$ so $du = 2x\,dx$ and then $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$. For $\displaystyle\int \frac{1}{x^2+2}\,dx$, use Formula 10 with $a = \sqrt{2}$. So $\displaystyle\int \frac{1}{x^2+2}\,dx = \int \frac{1}{x^2+(\sqrt{2})^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$.

Thus, $I = \dfrac{1}{2}\ln(x^2+1) + \dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}} + C$.

27.
$$\begin{array}{r|l} & \quad\quad\quad\quad\quad x \\ x^2+4 & x^3 + 0x^2 + 0x + 4 \\ & x^3 \quad\quad\quad + 4x \\ \hline & \quad\quad\quad -4x + 4 \end{array}$$

By long division, $\dfrac{x^3+4}{x^2+4} = x + \dfrac{-4x+4}{x^2+4}$. Thus,

$$\begin{aligned}\int \frac{x^3+4}{x^2+4}\,dx &= \int\left(x + \frac{-4x+4}{x^2+4}\right)dx \\ &= \int\left(x - \frac{4x}{x^2+4} + \frac{4}{x^2+2^2}\right)dx = \frac{1}{2}x^2 - 4\cdot\frac{1}{2}\ln|x^2+4| + 4\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \\ &= \frac{1}{2}x^2 - 2\ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C\end{aligned}$$

30. Let $u = \sqrt{x+2}$. Then $x = u^2 - 2$, $dx = 2u\,du \quad \Rightarrow \quad I = \int \frac{dx}{x - \sqrt{x+2}} = \int \frac{2u\,du}{u^2 - 2 - u} = 2\int \frac{u\,du}{u^2 - u - 2}$ and $\frac{u}{u^2 - u - 2} = \frac{A}{u-2} + \frac{B}{u+1} \quad \Rightarrow \quad u = A(u+1) + B(u-2)$. Substituting -1 for u gives $-1 = -3B \quad \Leftrightarrow \quad B = \frac{1}{3}$ and substituting 2 for u gives $2 = 3A \quad \Leftrightarrow \quad A = \frac{2}{3}$. Thus,

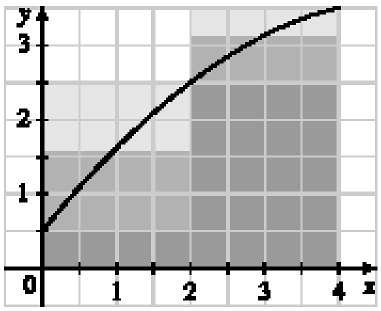
$$I = \frac{2}{3}\int \left[\frac{2}{u-2} + \frac{1}{u+1}\right] du = \frac{2}{3}(2\ln|u-2| + \ln|u+1|) + C$$
$$= \tfrac{2}{3}\left[2\ln\left|\sqrt{x+2} - 2\right| + \ln\left(\sqrt{x+2} + 1\right)\right] + C$$

1. (a) $\Delta x = (b-a)/n = (4-0)/2 = 2$

$$L_2 = \sum_{i=1}^{2} f(x_{i-1})\,\Delta x = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 2\,[f(0)+f(2)] = 2(0.5+2.5) = 6$$

$$R_2 = \sum_{i=1}^{2} f(x_i)\,\Delta x = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2\,[f(2)+f(4)] = 2(2.5+3.5) = 12$$

$$M_2 = \sum_{i=1}^{2} f(\overline{x}_i)\Delta x = f(\overline{x}_1)\cdot 2 + f(\overline{x}_2)\cdot 2 = 2\,[f(1)+f(3)] \approx 2(1.6+3.2) = 9.6$$

(b) 

L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I. See the solution to Exercise 37 for a proof of the fact that if f is concave down on $[a,b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)\,dx$.

(c) $T_2 = \left(\frac{1}{2}\,\Delta x\right)[f(x_0) + 2f(x_1) + f(x_2)] = \frac{2}{2}[f(0) + 2f(2) + f(4)] = 0.5 + 2(2.5) + 3.5 = 9.$
This approximation is an underestimate, since the graph is concave down. Thus, $T_2 = 9 < I$. See the solution to Exercise 37 for a general proof of this conclusion.

(d) For any n, we will have $L_n < T_n < I < M_n < R_n$.