PROBLEM SET 9
EXERCISES ON APPLYING THE DEFINITE INTEGRALL

PROBLEMS: 3, 5, 6, 11, 20, 22

Problem 3

We must be careful to do everything in yards. The density function is given by

$$\rho(x) = \frac{30 - x}{3}$$

where x is the distance from the goal line in yards. We want the number of oranges in the first 30 yards, given that the field is 160/3 yards wide. This will just be

$$\frac{160}{3}\int_0^{30} \frac{30-x}{3}\,dx = \frac{160}{9}\int_0^{30} 30 - x\,dx = \frac{160}{9}\left(30x - \frac{x^2}{2}\right)\Big|_0^{30} = \frac{160}{9}((900-450)-(0-0)) = 8000$$

So there will be 8000 oranges in that space.

Problem 5

The density is given by

$$\rho(h) = \frac{4}{h+10}$$

It is dependent only on height, so to compute the number of raisins in a box that is 5 cm deep tall, and 16 cm wide we only need to integrate over the height of the box. So the number of raisins will be

$$5 * 16 * \int_0^{25} \frac{4}{h+10}\,dx = 500\ln(h+10)|_0^{25} = 320(\ln 35 - \ln 10) = 320\ln(3.5) \approx 401$$

So there are about 401 raisins in the box.

Problem 6

Again, we have a density function given by

$$\rho(r) = \frac{1010}{\pi\left(r^2+1\right)^2}$$

For this we must consider integrating rings ranging from radius 0 to radius 10 (diameter 20). The length of a ring is just the circumfrence of a disk of the same radius which will be $2\pi r$, so our number of holes will just be

$$2\pi\int_0^{10} \rho(r) r\,dr = 1010\int_0^{10} \frac{2r}{\left(r^2+1\right)^2}\,dr = 1010\left(\frac{-1}{r^2+1}\right)\Big|_0^{10} = 1010\left((1 - \frac{1}{101}\right) = 1010\frac{100}{101} = 1000$$

So we have about 1000 holes.

Problem 11

Part A. Similar to what we did in number 6, the integral that gives the number of organisms in the dish will just be

$$2\pi \int_0^9 x f(x)\, dx$$

because this is the number of organisms along each ring, and we integrate over all the rings to form a disk. If we take that

$$f(x) = 100e^{-x^2}$$

then we get that our number of organisms will be

$$-100\pi \int_0^9 -2xe^{-x^2} = -100\pi\ e^{-x^2}\Big|_0^{18} = 100\pi\left(1 - e^{-81}\right) \approx 314$$

Part B. So you would slice it in slices parallel to the given strip of nutrients that runs along the diamater. This is because the two strips are a constant distance apart if they are parallel, so the density of organisms along these strips is uniform. What we can do for this is break it up into 180 slices, with 90 on each side of the strip in the middle. By symmetry we can just compute the number of organisms in one half and then double it, so we'll just deal with 90 slices, each of which is a centimeter wide. Given that the density of organisms a distance x from the strip is $f(x)$, we can estimate the number of organisms in the i^{th} slice by first computing the area of the ith slice. The width of each slice is just (1/10) of a centimeter, so we just need the length which will just be $l = 2\sqrt{9^2 - i^2}$ from the pythagorean theorem. Its true that these slices aren't exact rectangles, but if we make them skinny enough, the fact that the ends of it are curved doesn't change the area by a noticeable amount. Thus our estimate for the number of organisms would be

$$N_i = \frac{1}{10} 2\sqrt{9^2 - i^2} f(i)$$

Now if we want to find a riemann sum that represents the total number of organisms, we just need to sum over all the slices, remembering that we need to only sum 90 of them and then double it. This will yield

$$N = 2\sum_{i=0}^{89} \frac{1}{10} 2\sqrt{9^2 - i^2} f(i)$$

Now if we instead want an integral, we just need to take the limit in which the width of our slices goes to 0, so the width (1/10) will turn into a dx and our sum into an integral. Thus will give us

$$N = 2\int_0^9 2\sqrt{9^2 - x^2} f(x)\, dx$$

Problem 20

We have a 10 foot tall pole, with insects around it given by a density function of

$$\rho(r) = \frac{1.3}{\pi\,(r+1)}$$

Part A. For this we want to deal with insects that are a distance 5 feet from the pole, which extends down into the ground. Thus their height doesn't matter, just the distance from the center. So we'll integrate rings out until we have a radius of 5 feet, and multiply by the height of 10 feet. Thus we will have a number of insects equal to

$$20\pi \int_0^5 r\rho(r)\,dr = 26\int_0^5 \frac{r}{r+1}26\left(r-\ln(r+1)\right)\Big|_0^5 = 26\left(5-\ln(6)-0+\ln(1)\right) = 26\left(5-ln(6)\right) \approx 80$$

So there's 80 or so insects around the pole under 10 feet within 5 feet of the pole.

Part B. Now we are dealing with above the pole, so we must integrate spherical half-shells out to a radius of 5 feet. The important difference is that the pole stop, so it matters the distance from the top of the pole. So integrating shells out to a radius of 5 feet we will get

$$2\pi \int_0^5 r^2\rho(r)\,dr = 2.6\int_0^5 \frac{r^2}{r+1}\,dr = 2.6\left(\frac{(r+1)^2}{2} - 2\,(r+1)\ln(r+1)\right)\Bigg|_0^5 = 2.6\left(6+\ln 6+\frac{3}{2}\right) \approx 24$$

So there's about 24 flies in this region.

Problem 22

So the garlic varies with the distance of the center and the density function goes like

$$g(r) = \frac{r}{\left(r^3+2\right)^2}$$

We know that the pizza is 14 inches in diamater, so the radius is 7 inches. We cut six slices and want to find out the amount of garlic on one slice, so let's compute the total amount and then divide by 6. The density is a function of distance from the center, so lets integrate rings out to a radius of 7 inches. This will give us the amount of garlic as

$$2\pi\int_0^7 rg(r) = 2\pi\int_0^7 \frac{r^2}{\left(r^3+2\right)^2} = 2\pi\left(-\frac{1}{3\left(r^3+2\right)}\right)\Bigg|_0^7 = 2\pi\left(\frac{1}{6}-\frac{1}{3\left(7^3+2\right)}\right) = \frac{343\pi}{1035} \approx 1.04$$

So there's 1.04 ounces of garlic on the entire pie, to get the amount on a slice we divide by 6 to get

$$\frac{343\pi}{6210} \approx .17$$

So there's about .17 ounces on the slice.

1. $y = 2 - 3x \quad \Rightarrow \quad L = \int_{-2}^{1} \sqrt{1 + (dy/dx)^2}\, dx = \int_{-2}^{1} \sqrt{1 + (-3)^2}\, dx = \sqrt{10}\,[1 - (-2)] = 3\sqrt{10}.$

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$$L = [\text{distance from } (-2, 8) \text{ to } (1, -1)] = \sqrt{[1 - (-2)]^2 + [(-1) - 8]^2} = \sqrt{90} = 3\sqrt{10}$$

1. $g_{\text{ave}} = \frac{1}{\frac{\pi}{2} - 0} \int_0^{\pi/2} \cos x\, dx = \frac{2}{\pi}[\sin x]_0^{\pi/2} = \frac{2}{\pi}(1 - 0) = \frac{2}{\pi}$