3. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note after Example 4.)

(b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

4. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

13. $f(x) = xe^{-x}$ is continuous and positive on $[1, \infty)$. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x) < 0$ for $x > 1$, so f is decreasing on $[1, \infty)$. Thus, the Integral Test applies.

$$\int_1^\infty xe^{-x}\,dx = \lim_{b \to \infty} \int_1^b xe^{-x}\,dx = \lim_{b \to \infty} \left[-xe^{-x} - e^{-x}\right]_1^b \quad \text{[by parts]}$$

$$= \lim_{b \to \infty} \left[-be^{-b} - e^{-b} + e^{-1} + e^{-1}\right] = 2/e$$

since $\lim_{b \to \infty} be^{-b} = \lim_{b \to \infty} (b/e^b) \overset{\text{H}}{=} \lim_{b \to \infty} (1/e^b) = 0$ and $\lim_{b \to \infty} e^{-b} = 0$. Thus, $\sum_{n=1}^\infty ne^{-n}$ converges.

17. $\dfrac{\cos^2 n}{n^2 + 1} \le \dfrac{1}{n^2 + 1} < \dfrac{1}{n^2}$, so the series $\sum\limits_{n=1}^\infty \dfrac{\cos^2 n}{n^2 + 1}$ converges by comparison with the p-series $\sum\limits_{n=1}^\infty \dfrac{1}{n^2}$ $(p = 2 > 1)$.

21. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2 + 1}}$ and $b_n = \dfrac{1}{n}$:

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n}{\sqrt{n^2 + 1}} = \lim_{n \to \infty} \frac{1}{\sqrt{1 + (1/n^2)}} = 1 > 0.$$ Since the harmonic series $\sum\limits_{n=1}^\infty \dfrac{1}{n}$ diverges, so does $\sum\limits_{n=1}^\infty \dfrac{1}{\sqrt{n^2 + 1}}$.

28. (a) $f(x) = 1/x^4$ is positive and continuous and $f'(x) = -4/x^5$ is negative for $x > 0$, and so the Integral Test applies.

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \approx s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{10^4} \approx 1.082037.$$

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^4}\, dx = \lim_{t \to \infty} \left[\frac{1}{-3x^3} \right]_{10}^{t} = \lim_{t \to \infty} \left(-\frac{1}{3t^3} + \frac{1}{3\,(10)^3} \right) = \frac{1}{3000}, \text{ so the error is at most } 0.000\overline{3}.$$

(b) $s_{10} + \int_{11}^{\infty} \frac{1}{x^4}\, dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^4}\, dx \quad \Rightarrow \quad s_{10} + \frac{1}{3(11)^3} \leq s \leq s_{10} + \frac{1}{3(10)^3} \quad \Rightarrow$

$1.082037 + 0.000250 = 1.082287 \leq s \leq 1.082037 + 0.000333 = 1.082370$, so we get $s \approx 1.08233$ with error ≤ 0.00005.

(c) $R_n \leq \int_n^{\infty} \frac{1}{x^4}\, dx = \frac{1}{3n^3}$. So $R_n < 0.00001 \quad \Rightarrow \quad \frac{1}{3n^3} < \frac{1}{10^5} \quad \Rightarrow \quad 3n^3 > 10^5 \quad \Rightarrow \quad n > \sqrt[3]{(10)^5/3} \approx 32.2$,

that is, for $n > 32$.

32. $f(x) = \dfrac{1}{x(\ln x)^2}$ is positive and continuous and $f'(x) = -\dfrac{\ln x + 2}{x^2(\ln x)^3}$ is negative for $x > 1$, so the Integral Test applies.

Using (3), we need $0.01 > \int_n^{\infty} \dfrac{dx}{x(\ln x)^2} = \lim_{t \to \infty} \left[\dfrac{-1}{\ln x} \right]_n^{t} = \dfrac{1}{\ln n}$. This is true for $n > e^{100}$, so we would have to take this many terms, which would be problematic because $e^{100} \approx 2.7 \times 10^{43}$.

34. $\displaystyle\sum_{n=1}^{10} \frac{n}{(n+1)3^n} = \frac{1}{6} + \frac{2}{27} + \frac{3}{108} + \cdots + \frac{10}{649{,}539} \approx 0.283597.$ Now $\dfrac{n}{(n+1)3^n} < \dfrac{n}{n \cdot 3^n} = \dfrac{1}{3^n}$, so the error is

$$R_{10} \leq T_{10} = \sum_{n=11}^{\infty} \frac{1}{3^n} = \frac{1/3^{11}}{1 - 1/3} \approx 0.0000085.$$

1. (a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges if $0 < b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$. (This is the Alternating Series Test.)

(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$ and the size of the error is smaller than b_{n+1}; that is, $|R_n| \le b_{n+1}$. (This is the Alternating Series Estimation Theorem.)