2. (a) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

20. The series $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ has positive terms and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\right] = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\cdot\frac{1}{2} = \frac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.

24. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim_{n\to\infty}\frac{2^n}{n^4} = \infty$, so $\lim_{n\to\infty}(-1)^{n-1}\frac{2^n}{n^4}$ does not exist.

30. Use the Ratio Test with the series $\frac{2}{5}+\frac{2\cdot 6}{5\cdot 8}+\frac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11}+\frac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14}+\cdots = \sum_{n=1}^{\infty}\frac{2\cdot 6\cdot 10\cdot 14\cdot\cdots\cdot(4n-2)}{5\cdot 8\cdot 11\cdot 14\cdot\cdots\cdot(3n+2)}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2\cdot 6\cdot 10\cdot\cdots\cdot(4n-2)[4(n+1)-2]}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)[3(n+1)+2]}\cdot\frac{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}{2\cdot 6\cdot 10\cdot\cdots\cdot(4n-2)}\right| \\ &= \lim_{n\to\infty}\frac{4n+2}{3n+5} = \frac{4}{3} > 1,\end{aligned}$$

so the given series is divergent.

34. We use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2/[k(n+1)]!}{(n!)^2/(kn)!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{[k(n+1)]\,[k(n+1)-1]\cdots[kn+1]}\right|$$

Now if $k=1$, then this is equal to $\lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+1)}\right| = \infty$, so the series diverges; if $k=2$, the limit is $\lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| = \frac{1}{4} < 1$, so the series converges, and if $k>2$, then the highest power of n in the denominator is larger than 2, and so the limit is 0, indicating convergence. So the series converges for $k \geq 2$.

36. (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[4(n+1)]!\,[1103+26{,}390(n+1)]}{[(n+1)!]^4\,396^{4(n+1)}}\cdot\frac{(n!)^4\,396^{4n}}{(4n)!\,(1103+26{,}390n)}\right|$

$$= \lim_{n\to\infty}\frac{(4n+4)(4n+3)(4n+2)(4n+1)(26{,}390n+27{,}493)}{(n+1)^4\,396^4\,(26{,}390n+1103)}$$

$$= \frac{4^4}{396^4} = \frac{1}{99^4} < 1,$$

so by the Ratio Test, the series $\sum_{n=0}^{\infty}\frac{(4n)!\,(1103+26{,}390n)}{(n!)^4\,396^{4n}}$ converges.

(b) $\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!\,(1103+26{,}390n)}{(n!)^4\,396^{4n}}$

With the first term ($n=0$), $\frac{1}{\pi}\approx\frac{2\sqrt{2}}{9801}\cdot\frac{1103}{1} \Rightarrow \pi\approx 3.141\,592\,73$, so we get 6 correct decimal places of π, which is $3.141\,592\,653\,589\,793\,238$ to 18 decimal places.

With the second term ($n=1$), $\frac{1}{\pi}\approx\frac{2\sqrt{2}}{9801}\left(\frac{1103}{1}+\frac{4!\,(1103+26{,}390)}{396^4}\right) \Rightarrow \pi\approx 3.141\,592\,653\,589\,793\,878$, so we get 15 correct decimal places of π.

1. A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where x is a variable and the c_n's are constants called the coefficients of the series.
More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ is called a power series in $(x-a)$ or a power series centered at a or a power series about a, where a is a constant.

2. (a) Given the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, the radius of convergence is:
(i) 0 if the series converges only when $x = a$
(ii) ∞ if the series converges for all x, or
(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.
In most cases, R can be found by using the Ratio Test.

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.