7. If $a_n = \dfrac{x^n}{n!}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)!}\cdot\dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x}{n+1}\right| = |x|\lim\limits_{n\to\infty}\dfrac{1}{n+1} = |x|\cdot 0 = 0 < 1$ for *all* real x. So, by the Ratio Test, $R = \infty$, and $I = (-\infty, \infty)$.

8. If $a_n = \dfrac{x^n}{n3^n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)3^{n+1}}\cdot\dfrac{n3^n}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{xn}{(n+1)3}\right| = \dfrac{|x|}{3}\lim\limits_{n\to\infty}\dfrac{n}{n+1} = \dfrac{|x|}{3}$. By the Ratio Test, the series converges when $\dfrac{|x|}{3} < 1 \quad\Leftrightarrow\quad |x| < 3$, so $R = 3$. When $x = -3$, the series is the alternating harmonic series, which converges by the Alternating Series Test. When $x = 3$, it is the harmonic series, which diverges. Thus, $I = [-3, 3)$.

10. $a_n = \dfrac{x^n}{5^n n^5}$, so $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{5^{n+1}(n+1)^5}\cdot\dfrac{5^n n^5}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{5}\left(\dfrac{n}{n+1}\right)^5 = \dfrac{|x|}{5}$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty}\dfrac{x^n}{5^n n^5}$ converges when $\dfrac{|x|}{5} < 1 \quad\Leftrightarrow\quad |x| < 5$, so $R = 5$. When $x = -5$, we get the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^5}$, which converges by the Alternating Series Test. When $x = 5$, we get the convergent p-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^5}$ $(p = 5 > 1)$. Thus, $I = [-5, 5]$.

18. $a_n = \dfrac{n^2x^n}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)} = \dfrac{n^2x^n}{2^n n!} = \dfrac{nx^n}{2^n(n-1)!}$, so

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)\,|x|^{n+1}}{2^{n+1}n!}\cdot\dfrac{2^n(n-1)!}{n\,|x|^n} = \lim\limits_{n\to\infty}\dfrac{n+1}{n^2}\dfrac{|x|}{2} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

20. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \le x < 4$ and diverges for at least $x \ge 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$; that is, $\sum c_n(-3^n)$ is convergent.

(d) It diverges when $x = -9$; that is, $\sum c_n(-9)^n = \sum(-1)^n c_n 9^n$ is divergent.

21. If $a_n = \frac{(n!)^k}{(kn)!} x^n$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{[(n+1)!]^k \,(kn)!}{(n!)^k \,[k(n+1)]!} |x| = \lim_{n\to\infty} \frac{(n+1)^k}{(kn+k)(kn+k-1)\cdots(kn+2)(kn+1)} |x|$$

$$= \lim_{n\to\infty} \left[\frac{(n+1)}{(kn+1)} \frac{(n+1)}{(kn+2)} \cdots \frac{(n+1)}{(kn+k)} \right] |x|$$

$$= \lim_{n\to\infty} \left[\frac{n+1}{kn+1} \right] \lim_{n\to\infty} \left[\frac{n+1}{kn+2} \right] \cdots \lim_{n\to\infty} \left[\frac{n+1}{kn+k} \right] |x|$$

$$= \left(\frac{1}{k} \right)^k |x| < 1 \quad \Leftrightarrow \quad |x| < k^k$$ for convergence, and the radius of convergence is $R = k^k$.

27. For $2 < x < 3$, $\sum c_n x^n$ diverges and $\sum d_n x^n$ converges. By Exercise 8.2.51, $\sum (c_n + d_n)\, x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum (c_n + d_n)\, x^n$ is 2.

1. If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 10, then $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ also has radius of convergence 10 by Theorem 2.