

4.  $f(x) = \frac{3}{1-x^4} = 3\left(\frac{1}{1-x^4}\right) = 3(1+x^4+x^8+x^{12}+\dots) = 3\sum_{n=0}^{\infty}(x^4)^n = \sum_{n=0}^{\infty}3x^{4n}$

with  $|x^4| < 1 \iff |x| < 1$ , so  $R = 1$  and  $I = (-1, 1)$ .

[Note that  $3\sum_{n=0}^{\infty}(x^4)^n$  converges  $\iff \sum_{n=0}^{\infty}(x^4)^n$  converges, so the appropriate condition [from equation (1)] is  $|x^4| < 1$ .]

8.  $f(x) = \frac{x}{4x+1} = x \cdot \frac{1}{1-(-4x)} = x\sum_{n=0}^{\infty}(-4x)^n = \sum_{n=0}^{\infty}(-1)^n 2^{2n} x^{n+1}$ . The series converges when  $|-4x| < 1$ ; that is,

when  $|x| < \frac{1}{4}$ , so  $I = (-\frac{1}{4}, \frac{1}{4})$ .

12. (a)  $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n$  [geometric series with  $R = 1$ ], so

$$f(x) = \ln(1+x) = \int \frac{dx}{1+x} = \int \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

[ $C = 0$  since  $f(0) = \ln 1 = 0$ ], with  $R = 1$

(b)  $f(x) = x \ln(1+x) = x \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \right]$  [by part (a)]  $= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n} = \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n-1}$  with  $R = 1$ .

(c)  $f(x) = \ln(x^2+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x^2)^n}{n}$  [by part (a)]  $= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n}$  with  $R = 1$ .

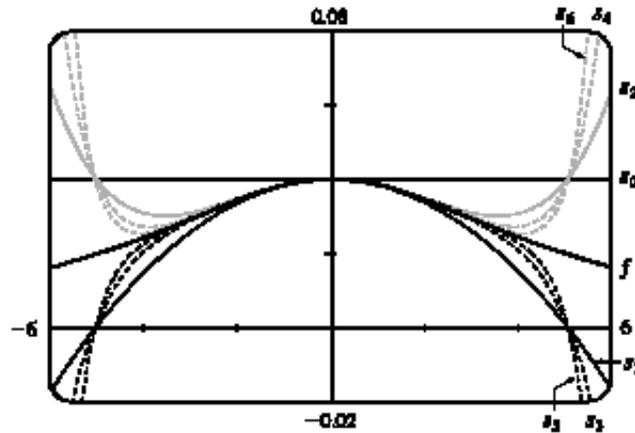
16. From Example 7,  $g(x) = \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ . Thus,

$$f(x) = \arctan(x/3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)} x^{2n+1} \text{ for } \left| \frac{x}{3} \right| < 1 \iff |x| < 3, \text{ so } R = 3.$$

18.  $f(x) = \frac{1}{x^2 + 25} = \frac{1}{25} \left( \frac{1}{1 + x^2/25} \right) = \frac{1}{25} \left( \frac{1}{1 - (-x^2/25)} \right) = \frac{1}{25} \sum_{n=0}^{\infty} \left( -\frac{x^2}{25} \right)^n = \frac{1}{25} \sum_{n=0}^{\infty} (-1)^n \left( \frac{x}{5} \right)^{2n}.$

The series converges when  $|-x^2/25| < 1 \iff x^2 < 25 \iff |x| < 5$ , so  $R = 5$ .

The terms of the series are  $a_0 = \frac{1}{25}, a_1 = -\frac{x^2}{625}, a_2 = \frac{x^4}{15,625}, \dots$



As  $n$  increases,  $s_n(x)$  approximates  $f$  better on the interval of convergence, which is  $(-5, 5)$ .

22. By Example 6,  $\ln(1 - t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$  for  $|t| < 1$ , so  $\frac{\ln(1 - t)}{t} = -\sum_{n=1}^{\infty} \frac{t^{n-1}}{n}$  and  $\int \frac{\ln(1 - t)}{t} dt = C - \sum_{n=1}^{\infty} \frac{t^n}{n^2}.$

By Theorem 2,  $R = 1$ .

30.  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(2n)!}$  [the first term disappears], so

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1)x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}$$
 [substituting  $n + 1$  for  $n$ ]
$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \Rightarrow f''(x) + f(x) = 0.$$

33. (a)  $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(b) By Theorem 7.4.2, the only solution to the differential equation  $df(x)/dx = f(x)$  is  $f(x) = Ke^x$ , but  $f(0) = 1$ , so  $K = 1$  and  $f(x) = e^x$ .

*Or:* We could solve the equation  $df(x)/dx = f(x)$  as a separable differential equation.

2. (a) Using Formula 6, a power series expansion of  $f$  at 1 must have the form  $f(1) + f'(1)(x - 1) + \dots$ . Comparing to the given series,  $1.6 - 0.8(x - 1) + \dots$ , we must have  $f'(1) = -0.8$ . But from the graph,  $f'(1)$  is positive. Hence, the given series is *not* the Taylor series of  $f$  centered at 1.
- (b) A power series expansion of  $f$  at 2 must have the form  $f(2) + f'(2)(x - 2) + \frac{1}{2}f''(2)(x - 2)^2 + \dots$ . Comparing to the given series,  $2.8 + 0.5(x - 2) + 1.5(x - 2)^2 - 0.1(x - 2)^3 + \dots$ , we must have  $\frac{1}{2}f''(2) = 1.5$ ; that is,  $f''(2)$  is positive. But from the graph,  $f$  is concave downward near  $x = 2$ , so  $f''(2)$  must be negative. Hence, the given series is *not* the Taylor series of  $f$  centered at 2.