**7.**

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ |
|---|---|---|
| 0 | $e^{5x}$ | 1 |
| 1 | $5e^{5x}$ | 5 |
| 2 | $5^2e^{5x}$ | 25 |
| 3 | $5^3e^{5x}$ | 125 |
| 4 | $5^4e^{5x}$ | 625 |
| $\vdots$ | $\vdots$ | $\vdots$ |

$$e^{5x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{5^n}{n!}x^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{5^{n+1}\,|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n\,|x|^n}\right]$$

$$= \lim_{n\to\infty}\frac{5\,|x|}{n+1} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.$$

**15.**

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(9)$ |
|---|---|---|
| 0 | $x^{-1/2}$ | $\frac{1}{3}$ |
| 1 | $-\frac{1}{2}x^{-3/2}$ | $-\frac{1}{2}\cdot\frac{1}{3^3}$ |
| 2 | $\frac{3}{4}x^{-5/2}$ | $-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\frac{1}{3^5}$ |
| 3 | $-\frac{15}{8}x^{-7/2}$ | $-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\frac{1}{3^7}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

$$\frac{1}{\sqrt{x}} = \frac{1}{3} - \frac{1}{2\cdot 3^3}(x-9) + \frac{3}{2^2\cdot 3^5}\frac{(x-9)^2}{2!} - \frac{3\cdot 5}{2^3\cdot 3^7}\frac{(x-9)^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot 3^{2n+1}\cdot n!}(x-9)^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)[2(n+1)-1]\,|x-9|^{n+1}}{2^{n+1}\cdot 3^{[2(n+1)+1]}\cdot(n+1)!}\cdot\frac{2^n\cdot 3^{2n+1}\cdot n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,|x-9|^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(2n+1)\,|x-9|}{2\cdot 3^2(n+1)}\right] = \frac{1}{9}\,|x-9| < 1$$

for convergence, so $|x-9| < 9$ and $R = 9$.

**22.** $\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \quad\Rightarrow\quad f(x) = \sin(x^4) = \sum_{n=0}^{\infty}(-1)^n\frac{(x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{8n+4}, \quad R = \infty$

**27.**

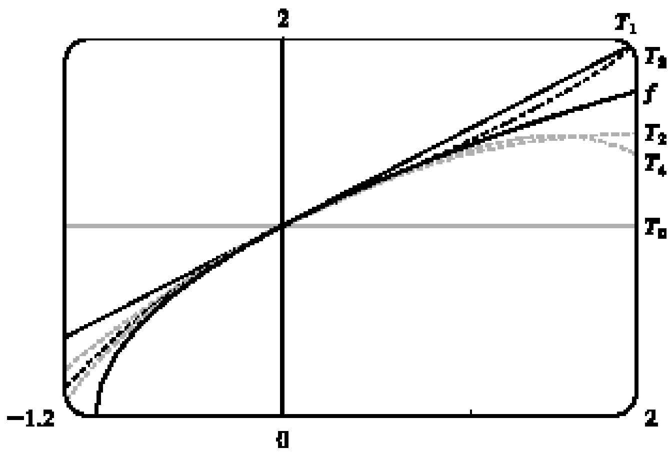
| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ |
|---|---|---|
| 0 | $(1+x)^{1/2}$ | 1 |
| 1 | $\frac{1}{2}(1+x)^{-1/2}$ | $\frac{1}{2}$ |
| 2 | $-\frac{1}{4}(1+x)^{-3/2}$ | $-\frac{1}{4}$ |
| 3 | $\frac{3}{8}(1+x)^{-5/2}$ | $\frac{3}{8}$ |
| 4 | $-\frac{15}{16}(1+x)^{-7/2}$ | $-\frac{15}{16}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

So $f^{(n)}(0) = \dfrac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n}$ for $n \geq 2$, and $\sqrt{1+x} = 1 + \dfrac{x}{2} + \displaystyle\sum_{n=2}^{\infty} \dfrac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!} x^n$.

If $a_n = \dfrac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!} x^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)(2n-1)x^{n+1}}{2^{n+1}(n+1)!} \cdot \frac{2^n n!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)x^n}\right|$$

$$= \frac{|x|}{2} \lim_{n\to\infty} \frac{2n-1}{n+1} = \frac{|x|}{2} \cdot 2 = |x| < 1 \text{ for convergence, so } R = 1.$$




Notice that, as $n$ increases, $T_n(x)$ becomes a better approximation to $f(x)$ for $-1 < x < 1$.

**32.** $3^\circ = \dfrac{\pi}{60}$ radians and $\sin x = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$, so

$\sin\dfrac{\pi}{60} = \dfrac{\pi}{60} - \dfrac{\left(\frac{\pi}{60}\right)^3}{3!} + \dfrac{\left(\frac{\pi}{60}\right)^5}{5!} - \cdots = \dfrac{\pi}{60} - \dfrac{\pi^3}{1{,}296{,}000} + \dfrac{\pi^5}{93{,}312{,}000{,}000} - \cdots$. But $\dfrac{\pi^5}{93{,}312{,}000{,}000} < 10^{-8}$, so by the Alternating Series Estimation Theorem, $\sin\dfrac{\pi}{60} \approx \dfrac{\pi}{60} - \dfrac{\pi^3}{1{,}296{,}000} \approx 0.05234$.

**33.** $\cos x \overset{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad\Rightarrow\quad \cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} \quad\Rightarrow$

$x\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!} \quad\Rightarrow\quad \int x\cos(x^3)\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$, with $R = \infty$.

**42.** $$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{1-\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)}{1+x-\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\cdots\right)}$$
$$= \lim_{x\to 0} \frac{\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6-\cdots}{-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{1}{4!}x^4-\frac{1}{5!}x^5-\frac{1}{6!}x^6-\cdots}$$
$$= \lim_{x\to 0} \frac{\frac{1}{2!}-\frac{1}{4!}x^2+\frac{1}{6!}x^4-\cdots}{-\frac{1}{2!}-\frac{1}{3!}x-\frac{1}{4!}x^2-\frac{1}{5!}x^3-\frac{1}{6!}x^4-\cdots} = \frac{\frac{1}{2}-0}{-\frac{1}{2}-0} = -1$$

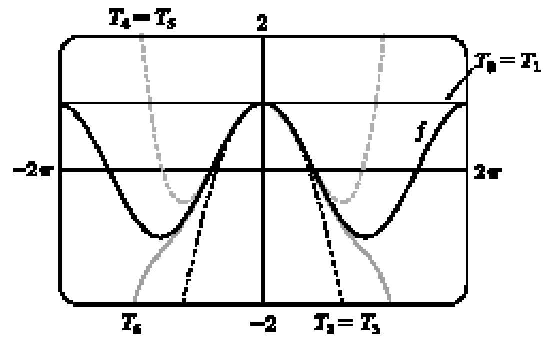
since power series are continuous functions.

**48.** From Example 6 in Section 8.6, we have $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$, $|x| < 1$. Therefore,

$$e^x \ln(1-x) = \left(1+x+\frac{1}{2}x^2+\cdots\right)\left(-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots\right)$$
$$= -x-\frac{1}{2}x^2-\frac{1}{3}x^3-x^2-\frac{1}{2}x^3-\frac{1}{2}x^3-\cdots = -x-\frac{3}{2}x^2-\frac{4}{3}x^3-\cdots,\ |x|<1$$

**1.** (a)

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ | $T_n(x)$ |
|---|---|---|---|
| 0 | $\cos x$ | 1 | $1$ |
| 1 | $-\sin x$ | 0 | $1$ |
| 2 | $-\cos x$ | $-1$ | $1-\frac{1}{2}x^2$ |
| 3 | $\sin x$ | 0 | $1-\frac{1}{2}x^2$ |
| 4 | $\cos x$ | 1 | $1-\frac{1}{2}x^2+\frac{1}{24}x^4$ |
| 5 | $-\sin x$ | 0 | $1-\frac{1}{2}x^2+\frac{1}{24}x^4$ |
| 6 | $-\cos x$ | $-1$ | $1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$ |




(b)

| $x$ | $f$ | $T_0=T_1$ | $T_2=T_3$ | $T_4=T_5$ | $T_6$ |
|---|---|---|---|---|---|
| $\frac{\pi}{4}$ | 0.7071 | 1 | 0.6916 | 0.7074 | 0.7071 |
| $\frac{\pi}{2}$ | 0 | 1 | $-0.2337$ | 0.0200 | $-0.0009$ |
| $\pi$ | $-1$ | 1 | $-3.9348$ | 0.1239 | $-1.2114$ |

(c) As $n$ increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.